Contrôle optimal d'équations différentielles avec - ou sans - mémoire

Xavier Dupuis

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J. Frédéric Bonnans	directeur
Ugo Boscain	examinateur
Jean-Baptiste Caillau	président
Piermarco Cannarsa	rapporteur
Jean-Michel Coron	rapporteur
Fabien Crauste	examinateur

Résumé

La thèse porte sur des problèmes de contrôle optimal où la dynamique est donnée par des équations différentielles avec mémoire. Pour ces problèmes d'optimisation, des conditions d'optimalité sont établies ; celles du second ordre constituent une part importante des résultats de la thèse. Dans le cas – sans mémoire – des équations différentielles ordinaires, les conditions d'optimalité standards sont renforcées en ne faisant intervenir que les multiplicateurs de Lagrange pour lesquels le principe de Pontryaguine est satisfait. Cette restriction à un sous-ensemble des multiplicateurs représente un défi dans l'établissement des conditions nécessaires et permet aux conditions suffisantes d'assurer l'optimalité locale dans un sens plus fort. Les conditions standards sont d'autre part étendues au cas – avec mémoire – des équations intégrales. Les contraintes pures sur l'état du problème précédent ont été conservées et nécessitent une étude spécifique à la dynamique intégrale. Une autre forme de mémoire dans l'équation d'état d'un problème de contrôle optimal provient d'un travail de modélisation avec l'optimisation thérapeutique comme application médicale en vue. La dynamique de populations de cellules cancéreuses sous l'action d'un traitement est ramenée à des équations différentielles à retards ; le comportement asymptotique en temps long du modèle structuré en âge est également étudié.

Mots-clefs

contrôle optimal, conditions d'optimalité, équations différentielles avec mémoire, dynamique de populations, application médicale

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Optimal control of differential equations with – or without – memory

Abstract

The thesis addresses optimal control problems where the dynamics is given by differential equations with memory. For these optimization problems, optimality conditions are provided; second order conditions constitute an important part of the results of the thesis. In the case – without memory – of ordinary differential equations, standard optimality conditions are strengthened by involving only the Lagrange multipliers for which Pontryagin's principle is satisfied. This restriction to a subset of multipliers represents a challenge in the establishment of necessary conditions and enables sufficient conditions to assure local optimality in a stronger sense. Standard conditions are on the other hand extended to the case – with memory – of integral equations. Pure state constraints of the previous problem have been kept and require a specific study due to the integral dynamics. Another form of memory in the state equation of an optimal control problem comes from a modeling work with therapeutic optimization as a medical application in view. Cancer cells populations dynamics under the action of a treatment is reduced to delay differential equations; the long time asymptotics of the age-structured model is also studied.

Keywords

optimal control, optimality conditions, differential equations with memory, population dynamics, medical application

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Introduction

1 Introduction générale

1.1 Contrôle optimal et équations différentielles avec mémoire

Le contrôle optimal correspond à la branche *optimisation* de la théorie mathématique du contrôle ; un problème de contrôle optimal est avant tout un problème d'optimisation dans des espaces de Banach. Une forme générale d'un tel problème, que l'on nomme ici problème abstrait d'optimisation, est la suivante : soient X et Y deux espaces de Banach, $K \subset Y$ un convexe fermé, $J \colon X \to \mathbb{R}$ et $G \colon X \to Y$; on considère alors le problème

$$\min_{x \in X} J(x) \quad \text{sous la contrainte} \quad G(x) \in K. \tag{P_A}$$

Dans un problème de contrôle optimal, l'espace X est celui des couples de fonctions (u, y) où u est un contrôle et y un état. Le contrôle et l'état sont liés par un système dynamique, qui s'apparente à une contrainte d'égalité, dans lequel le contrôle peut être choisi extérieurement alors que l'état en est la solution. Un tel problème est généralement équivalent à sa version réduite, où la variable d'optimisation est uniquement le contrôle u et où la fonction objectif J et la contrainte G portent sur u et l'état associé via le système dynamique $y[u]$.

Le cadre classique du contrôle optimal est celui des équations différentielles ordinaires : le système dynamique, que l'on appelle équation d'état, est de la forme

$$\dot{y}(t) = f\left(t, u(t), y(t)\right),$$

et par exemple $u \in L^\infty\left(0, T; \mathbb{R}^m\right)$, $y \in W^{1,\infty}\left(0, T; \mathbb{R}^n\right)$. Une telle équation peut être vue comme un cas particulier – sans mémoire – d'équations d'état que l'on qualifiera d'*équations différentielles avec mémoire* et qui s'écrivent

$$\dot{y}(t) = \left\langle f\left(t, u(\cdot), y(\cdot)\right), \nu_t \right\rangle,$$

où ν_t est pour tout $t \geq 0$ une mesure à support dans $[0, t]$. Le sens à donner à cette écriture est le suivant :

$$\left\langle f\left(t, u(\cdot), y(\cdot)\right), \nu_t \right\rangle := \int_{[0,t]} f\left(t, u(s), y(s)\right) \mathrm{d}\nu_t(s).$$

La mémoire dans l'équation à l'instant t est ainsi représentée par la mesure ν_t, le passé étant $[0, t]$. Le cadre des équations différentielles avec mémoire couvre un certain nombre de systèmes dynamiques de forme connue :

1. les équations différentielles ordinaires, comme annoncé, avec $\nu_t := \delta_t$. La masse de Dirac δ_t porte uniquement sur l'instant présent t ; ces équations n'ont pas de mémoire dans ce sens.

2. les équations différentielles à retard discret

$$\dot{y}(t) = f\left(t, u(t-\tau), y(t-\tau)\right),$$

avec $\nu_t := \delta_{t-\tau}$ si $t > \tau$, $\tau > 0$ fixe. La mémoire se concentre sur un instant, toujours à même distance dans le passé.

3. les équations différentielles à argument dévié

$$\dot{y}(t) = f\left(t, u(\theta_t), y(\theta_t)\right),$$

avec $\nu_t := \delta_{\theta_t}$, $0 \leq \theta_t \leq t$ variable. La mémoire est encore ponctuelle.

4. les équations différentielles à retard distribué

$$\dot{y}(t) = \int_{t-\tau}^{t} f\left(t, u(s), y(s)\right) \mathrm{d}s,$$

avec $\nu_t := \chi_{(t-\tau,t)}\mathcal{L}^1$ si $t > \tau$, $\tau > 0$ et $\mathcal{L}^1$ désignant la mesure de Lebesgue. La mémoire prend ici en compte une portion de longueur fixe du passé.

5. les équations intégro-différentielles de type Volterra

$$\dot{y}(t) = \int_{0}^{t} f\left(t, u(s), y(s)\right) \mathrm{d}s,$$

avec $\nu_t := \chi_{(0,t)}\mathcal{L}^1$. La mémoire prend en compte tout le passé.

Ce cadre peut être étendu en considérant pour tout t des mesures ν_t à valeurs dans $\mathbb{R}^k$. On peut alors ajouter aux exemples précédents

6. les équations différentielles obtenues en sommant les membres de droite des équations citées ci-dessus ; en particulier les équations intégrales de type Volterra

$$y(t) = y(0) + \int_{0}^{t} f\left(t, u(s), y(s)\right) \mathrm{d}s,$$

avec $\nu_t^1 := \delta_t$, $f^1 := f$, $\nu_t^2 := \chi_{(0,t)}\mathcal{L}^1$, $f^2 := \frac{\partial f}{\partial t}$.

Outre le système dynamique, les autres éléments d'un problème de contrôle optimal, en tant que problème d'optimisation, sont une fonctionnelle à minimiser – ou maximiser – sur l'ensemble des trajectoires et éventuellement des contraintes que doivent satisfaire les trajectoires pour être admissibles. Une forme générale de fonction objectif en contrôle optimal est celle de Bolza :

$$J(u,y) := \int_0^T \ell(t, u(t), y(t))\mathrm{d}t + \phi(y(0), y(T)).$$

En introduisant une variable d'état supplémentaire, cette fonction coût peut toujours se mettre sous la forme de Mayer :

$$J(u,y) := \phi(y(0), y(T)).$$

Les contraintes peuvent être imposées dans des espaces vectoriels de dimension finie ou infinie. Les contraintes terminales sur l'état

$$\Phi(y(0), y(T)) \in K$$

avec K un polyèdre, qui incluent également via l'introduction d'une variable d'état supplémentaire les contraintes isopérimétriques

$$\int_0^T h(t, u(t), y(t))\mathrm{d}t \leq L \quad \text{ou} \quad \int_0^T h(t, u(t), y(t))\mathrm{d}t = L,$$

le sont dans un espace de dimension finie. Les contraintes sur la trajectoire, mixtes sur le contrôle et l'état

$$c(t, u(t), y(t)) \leq 0$$

ou pures sur l'état

$$g(t, y(t)) \leq 0$$

à tout instant, sont quant à elles vues dans différents espaces de fonctions de dimension infinie.

Un problème de contrôle optimal d'équations différentielles avec mémoire dans sa généralité peut finalement se présenter ainsi :

$$\begin{aligned}
&\min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \phi(y(0), y(T)) \quad \text{sous les contraintes} \\
&\dot{y}(t) = \langle f\left(t, u(\cdot), y(\cdot)\right), \nu_t\rangle \quad t \in (0, T), \\
&\Phi(y(0), y(T)) \in K, \\
&c(t, u(t), y(t)) \leq 0 \quad t \in (0, T), \\
&g(t, y(t)) \leq 0 \quad t \in (0, T),
\end{aligned} \tag{P_C}$$

où $\mathcal{U} := L^\infty(0, T; \mathbb{R}^m)$, $\mathcal{Y} := W^{1,\infty}(0, T; \mathbb{R}^n)$.

1.2 Conditions d'optimalité

Les conditions d'optimalité pour un problème d'optimisation visent à en caractériser les solutions, autrement dit les extrémas. Les conditions nécessaires permettent d'isoler des candidats à l'optimalité, ou du moins d'en éliminer et de déduire des propriétés qualitatives et quantitatives sur les solutions ; les conditions suffisantes assurent quant à elles qu'un extremum potentiel est effectivement localement optimal. Les conditions du premier et du second ordre sont également à la base d'algorithmes de type méthode de Newton – pour résoudre numériquement le problème d'optimisation – et de la preuve de leur caractère bien posé. Les conditions du second ordre sont de plus étroitement liées à l'analyse de sensibilité de problèmes d'optimisation soumis à des perturbations [25].

Pour un problème abstrait d'optimisation (P_A), les conditions d'optimalité sont formulées à l'aide du *Lagrangien*

$$L[\lambda](x) := J(x) + \langle \lambda, G(x) \rangle\,,$$

où $\lambda \in Y^*$ – l'espace dual de Y – et $\langle \cdot, \cdot \rangle$ désigne le produit de dualité. Les conditions nécessaires du premier ordre expriment, pour un minimum local $\bar{x}$ satisfaisant une certaine condition de régularité – dite condition de qualification – de la contrainte, l'existence de *multiplicateurs de Lagrange* [92] :

$$\Lambda := \left\{ \lambda \in N_K(G(\bar{x})) \,:\, L[\lambda]'(\bar{x}) = 0 \right\} \neq \emptyset,$$

avec $N_K(\cdot)$ le cône normal à K au point considéré. Une formulation des conditions nécessaires du second ordre est la suivante [11] : sous la même condition de régularité, pour tout $v \in C^R(\bar{x})$, il existe $\lambda \in \Lambda$ tel que

$$L[\lambda]''(\bar{x})(v, v) \geq 0$$

L'ensemble $C^R(\bar{x})$ est un sous-cône, constitué de directions radiales pour la contrainte linéarisée, du cône critique $C(\bar{x})$ qui intervient dans les conditions suffisantes du second ordre [67] : si $\bar{x}$ est admissible et régulier et s'il existe $\alpha > 0$ tel que pour tout $v \in C(\bar{x})$, il existe $\lambda \in \Lambda$ tel que

$$L[\lambda]''(\bar{x})(v,v) \geq \alpha \|v\|^2,$$

alors $\bar{x}$ est un minimum local.

Pour un problème de contrôle optimal d'équations différentielles ordinaires, sans mémoire, c'est-à-dire de la forme (P_C) avec $\nu_t := \delta_t$ pour tout t, les conditions d'optimalité s'expriment en fonction du *Hamiltonien*

$$H[p](t,u,y) := pf(t,u,y).$$

En effet, en notant p la variable duale relative à la contrainte dynamique, le Lagrangien de ce problème d'optimisation fait intervenir

$$\int_0^T H[p(t)](t,u(t),y(t))\mathrm{d}t.$$

Les conditions nécessaires du premier ordre pour un minimum local $(\bar{u},\bar{y})$ se traduisent alors, dans le cas sans contrainte, par l'existence d'un *état adjoint* p solution du problème aux deux bouts

$$\begin{cases} -\dot{p}(t) = D_y H[p(t)](t,\bar{u}(t),\bar{y}(t)) \\ p(T) = D_{y_T}\phi(\bar{y}(0),\bar{y}(T)) \\ -p(0) = D_{y_0}\phi(\bar{y}(0),\bar{y}(T)) \end{cases}$$

et pour lequel le Hamiltonien satisfait, pour presque t, la condition de stationnarité

$$D_u H[p(t)](t,\bar{u}(t),\bar{y}(t)) = 0.$$

L'état adjoint est ici unique, ce qui n'est plus le cas en général en présence de contraintes.

Les conditions du second ordre se déduisent de façon similaire, avec les complications suivantes : d'une part, pour des problèmes avec contraintes pures sur l'état, le cône des directions critiques et radiales des conditions nécessaires est significativement plus petit que celui des directions seulement critiques des conditions suffisantes. Pour réduire l'écart entre ces conditions, il faut soit établir celles nécessaires directement sur le cône critique : un effet *enveloppe-like*, découvert par Kawasaki pour les problèmes d'optimisation avec une infinité de contraintes d'inégalité, apparaît alors [34, 58] ; soit reformuler le problème suivant une approche par *réduction.* D'autre part, pour tout problème de contrôle optimal, les conditions suffisantes ne peuvent être satisfaites pour la norme $\|\cdot\|_\infty$; il faut en partie travailler avec la norme $\|\cdot\|_2$, d'où la *two-norm discrepancy* [66].

Outre les conditions d'optimalité héritées de la théorie de l'optimisation dans des espaces de Banach, il existe avec le *principe de Pontryaguine* des conditions nécessaires du premier ordre plus fortes, spécifiques au contrôle optimal [78]. Elles s'appliquent à une solution $(\bar{u},\bar{y})$ pour une notion plus forte d'optimalité locale et fournissent l'existence d'un état adjoint p pour le lequel, dans le cas sans contrainte, le Hamiltonien satisfait la condition de minimisation

$$H[p(t)](t,\bar{u}(t),\bar{y}(t)) \leq H[p(t)](t,u,\bar{y}(t)) \quad \text{pour tout } u \in \mathbb{R}^m,$$

pour presque tout t ; ce qui implique la condition de stationnarité.

Un grand intérêt – du point de vue de l'optimisation – du cadre introduit précédemment des équations différentielles avec mémoire, apparaît dans [27] et repose sur un théorème de désintégration [4]. Etant donnée une famille $(\nu_t)_t$ de mesures sur $[0,T]$ et $\mathcal{L}^1$ la mesure de Lebesgue sur $[0,T]$, soient $\gamma := \nu_t \otimes \mathcal{L}^1$ et ν la seconde marginale de γ ; il existe alors une famille de mesures $(\nu_s^*)_s$ sur $[0,T]$ telle que $\gamma = \nu \otimes \nu_s^*$. Ainsi, pour tout $\varphi \in L^1(\gamma)$,

$$\int_0^T \langle \varphi(t,\cdot), \nu_t \rangle \,\mathrm{d}t = \int_0^T \langle \varphi(\cdot,s), \nu_s^* \rangle \,\mathrm{d}\nu(s).$$

On peut alors définir le Hamiltonien *non local* – seulement en la variable p – du problème (P_C) dans le cas avec mémoire :

$$H[p](t,u,y) := \langle p(\cdot) f(\cdot,u,y), \nu_t^* \rangle \,.$$

Le Lagrangien de ce problème de contrôle optimal fait maintenant intervenir

$$\int_0^T p(t) \langle f(t,u(\cdot),y(\cdot)), \nu_t \rangle \,\mathrm{d}t = \int_0^T H[p](t,u(t),y(t))\mathrm{d}\nu(t).$$

En particulier les variables d'optimisation u et y y apparaissent toujours ponctuellement, d'où une possible extension des conditions d'optimalité présentées dans le cas sans mémoire. Puisque les mesures ν_t sont à support dans $[0,t]$, les mesures ν_s^* sont à support dans $[s,T]$ pour tout s ; on s'attend donc à ce que la dynamique adjointe,

$$-\mathrm{d}p(t) = D_y H[p](t,u(t),y(t))\mathrm{d}\nu(t)$$

dans le cas sans contrainte, soit à argument avancé. Reprenant les exemples initiaux d'équations différentielles avec mémoire, on obtient :

1. pour $\nu_t := \delta_t$, $\nu = \mathcal{L}^1$, $\nu_s^* = \delta_s$ et
$$H[p](t,u,y) = p(t)f(t,u,y).$$
On retrouve bien le Hamiltonien du cas sans mémoire.
2. pour $\nu_t := \delta_{t-\tau}$ si $t > \tau$, $\nu = \mathcal{L}^1$, $\nu_s^* = \delta_{s+\tau}$ si $s+\tau < T$ et
$$H[p](t,u,y) = \chi_{(0,T-\tau)}(t)p(t+\tau)f(t+\tau,u,y).$$
3. pour $\nu_t := \delta_{\theta_t}$, $\nu = \dot{\theta}^{-1}\mathcal{L}^1$, $\nu_s^* = \delta_{\theta_s^{-1}}$ si $s < \theta_T$ et
$$H[p](t,u,y) = \chi_{(0,\theta_T)}(t)p(\theta_t^{-1})f(\theta_t^{-1},u,y).$$
4. pour $\nu_t := \chi_{(t-\tau,t)}\mathcal{L}^1$ si $t > \tau$, $\nu = \mathcal{L}^1$, $\nu_s^* = \chi_{(s,s+\tau)\cap(\tau,T)}\mathcal{L}^1$ et
$$H[p](t,u,y) = \int_t^{t+\tau} \chi_{(\tau,T)}(s)p(s)f(s,u,y)\mathrm{d}s.$$
5. pour $\nu_t := \chi_{(0,t)}\mathcal{L}^1$, $\nu = \mathcal{L}^1$, $\nu_s^* = \chi_{(s,T)}\mathcal{L}^1$ et
$$H[p](t,u,y) = \int_t^T p(s)f(s,u,y)\mathrm{d}s.$$

2 Apports de la thèse

Cette thèse porte sur des problèmes de contrôle optimal de la forme (P_C), où la mémoire et les contraintes sont spécifiées. Les principaux apports concernent les conditions d'optimalité pour certains de ces problèmes – renforcement dans le cas sans mémoire, extension dans un cas avec mémoire – et une application médicale fournissant un exemple avec mémoire.

2.1 Conditions d'optimalité

Conditions d'optimalité sous forme Pontryaguine

Les Chapitres 1 et 2, qui constituent la Partie I de la thèse, présentent des conditions d'optimalité fortes pour le problème (P_C) sans mémoire et avec toutes les contraintes ; soit le problème

$$\begin{aligned} \min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \ & \phi(y(0),y(T)) \quad \text{sous les contraintes} \\ & \dot{y}(t) = f\,(t,u(t),y(t)) \quad t\in(0,T), \\ & \Phi(y(0),y(T)) \in K, \\ & c(t,u(t),y(t)) \leq 0 \quad t\in(0,T), \\ & g(t,y(t)) \leq 0 \quad t\in(0,T), \end{aligned} \tag{$P_{1,2}$}$$

où $\mathcal{U} := L^\infty(0,T;\mathbb{R}^m)$, $\mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n)$. Pour simplifier l'introduction, on fait l'hypothèse que toutes les données sont de classe C^∞ et que c et g sont scalaires.

Etant donnée une trajectoire admissible $(\bar{u},\bar{y})$, les multiplicateurs de Lagrange de ce problème sont les quadruplets $\lambda = (\beta,\Psi,\nu,\mu)$ tels que

- $\beta\in\mathbb{R}_+$ est associé à la fonction objectif ϕ ; on considère donc en fait des multiplicateurs *généralisés* pour simplifier les questions de qualification.
- Ψ est un vecteur de dimension finie du cône normal à K en $\Phi(\bar{y}(0),\bar{y}(T))$.
- ν est un élément du cône normal aux fonctions essentiellement bornées négatives en $c(\cdot,\bar{u}(\cdot),\bar{y}(\cdot))$; on suppose dans toute la suite qu'il existe $\gamma>0$ et $\bar{v}\in\mathcal{U}$ tels que, pour presque tout t,

  $$c(t,\bar{u}(t),\bar{y}(t)) + D_u c(t,\bar{u}(t),\bar{y}(t))\bar{v}(t) \leq -\gamma, \tag{1}$$

  ce qui permet d'exiger que ν soit une fonction essentiellement bornée.
- μ est un élément du cône normal aux fonctions continues négatives en $g(\cdot,\bar{y}(\cdot))$; c'est une mesure de Radon.
- il existe un état adjoint p^λ pour lequel le *Hamiltonien augmenté*

  $$H^a[p,\nu](t,u,y) := pf(t,u,y) + \nu c(t,u,y)$$

  satisfait, pour presque tout t, la condition de stationnarité

  $$D_u H^a[p^\lambda(t),\nu(t)](t,\bar{u}(t),\bar{y}(t)) = 0.$$

L'équation adjointe est définie dans l'espace de fonctions à variations bornées par

$$\begin{cases} -\mathrm{d}p(t) = D_y H^a[p(t),\nu(t)](t,\bar{u}(t),\bar{y}(t))\mathrm{d}t + \mathrm{d}\mu(t) D_y g(t,\bar{y}(t)) \\ \quad p(T) = D_{y_T}\Phi[\beta,\Psi](\bar{y}(0),\bar{y}(T)) \\ -p(0) = D_{y_0}\Phi[\beta,\Psi](\bar{y}(0),\bar{y}(T)) \end{cases}$$

où $\Phi[\beta,\Psi](y_0,y_T) := \beta\phi(y_0,y_T) + \Psi\Phi(y_0,y_T)$ est le *Lagrangien terminal.* On désigne par Λ_L l'ensemble des multiplicateurs de Lagrange généralisés. Les multiplicateurs de Pontryaguine généralisés sont ensuite définis comme les multiplicateurs $\lambda\in\Lambda_L$ pour lesquels, en plus de la condition de stationnarité du Hamiltonien augmenté, le Hamiltonien non augmenté $H[p](t,u,y) := pf(t,u,y)$ satisfait, pour presque tout t, la condition de minimisation

$$H[p^\lambda(t)](t,\bar{u}(t),\bar{y}(t)) \leq H[p^\lambda(t)](t,u,\bar{y}(t))$$

pour tout u dans l'adhérence de $\{c(t,\cdot,\bar{y}(t)) < 0\}$. Soit Λ_P l'ensemble des multiplicateurs de Pontryaguine généralisés ; par définition, $\Lambda_P \subset \Lambda_L$.

On appelle dans la thèse *conditions d'optimalité sous forme Pontryaguine* les conditions qui ne font intervenir que les multiplicateurs de Pontryaguine. Cette notion correspond au premier ordre – avec l'existence de multiplicateurs – au principe du minimum de Pontryaguine, et rejoint au second ordre celle de conditions quadratiques d'Osmolovskii [69, 73, 71, 72]. De telles conditions nécessaires constituent un renforcement des conditions classiques, sous forme Lagrange, et les conditions suffisantes sous forme Pontryaguine permettent d'assurer l'optimalité locale d'une trajectoire dans un sens plus fort. La principale nouveauté de la thèse dans cette direction est l'établissement de conditions du second ordre sous forme Pontryaguine pour des problèmes avec parmi d'autres des contraintes pures sur l'état. En effet, de telles contraintes sont exclues par l'hypothèse d'indépendance linéaire faite sur les contraintes mixtes par Osmolovskii dans ses travaux. D'autre part, l'approche présentée dans ce manuscrit est d'une certaine façon plus élementaire puisqu'elle n'utilise pas la théorie des γ-conditions.

Définition 1. Soit $(\bar{u},\bar{y})$ une trajectoire admissible. On dit que $(\bar{u},\bar{y})$ est un *minimum borné fort* si pour tout $R > 0$, il existe $\varepsilon > 0$ tel que

$$\phi(y(0),y(T)) \geq \phi(\bar{y}(0),\bar{y}(T))$$

pour toute trajectoire admissible (u,y) telle que $\|u\|_\infty \leq R$ et $\|y-\bar{y}\|_\infty \leq \varepsilon$.

On dit de plus qu'il y a *croissance quadratique* si pour tout $R > 0$, il existe $\varepsilon > 0$ et $\alpha > 0$ tels que

$$\phi(y(0),y(T)) - \phi(\bar{y}(0),\bar{y}(T)) \geq \alpha \left(|u-\bar{u}\|_2^2 + \|y-\bar{y}\|_\infty^2 \right)$$

pour toute trajectoire admissible (u,y) telle que $\|u\|_\infty \leq R$ et $\|y-\bar{y}\|_\infty \leq \varepsilon$.

Cette notion d'optimalité est plus forte que celle de minimum local – dit *minimum faible* – héritée de la théorie de l'optimisation dans des espaces de Banach ; une troisième notion d'optimalité locale, intermédiaire, est celle de *minimum de Pontryaguine* ; voir les Définitions 2.2.4 et 2.2.5. Un premier résultat du Chapitre 1 est le Théorème 1.3.1 :

Théorème 2. *Soit $(\bar{u},\bar{y})$ un minimum borné fort. On suppose que la condition rentrante* (1) *est satisfaite. Alors $\Lambda_P \neq \emptyset$.*

Ce résultat est obtenu dans la thèse pour un minimum de Pontryaguine, avec des données de classe C^1, voire un peu moins, et des contraintes mixtes et pures vectorielles. Il n'est pas nouveau en soi, voir par exemple [36], mais permet de mettre en place la méthode utilisée pour établir les conditions nécessaires du second ordre sous forme Pontryaguine. Cette méthode est celle de la *relaxation partielle* ou des sliding modes, employée par Dmitruk dans [36] suivant une idée de Milyutin. Etant donnés des contrôles $u^1,\dots,u^N$ fixés, on considère l'équation d'état partiellement relaxée

$$\dot{y}(t) = \left(1 - \sum_{i=1}^N \alpha^i(t)\right) f(t,u(t),y(t)) + \sum_{i=1}^N \alpha^i(t) f(t,u^i(t),y(t)) \quad t \in (0,T),$$

et le problème de contrôle optimal consistant à
– minimiser la même fonction objectif ϕ,
– sur l'ensemble des contrôles $u \in \mathcal{U}$, $\alpha^i \in L^\infty(0,T)$ et des états relaxés $y \in \mathcal{Y}$ associés,

– sous les mêmes contraintes, plus les contraintes sur les contrôles

$$\alpha^i(t) \geq 0 \quad t \in (0,T).$$

Pour un choix admissible de $u^1, \ldots, u^N$, le Théorème 1.3.7 valide l'introduction de ce problème :

Théorème 3. *Soit* $(\bar{u}, \bar{y})$ *un minimum borné fort et* $\bar{\alpha} := (0, \ldots, 0)$*. On suppose que la condition rentrante* (1) *est satisfaite et que le problème partiellement relaxé est qualifié. Alors* $(\bar{u}, \bar{y}, \bar{\alpha})$ *est un minimum faible de ce problème.*

Il s'agit d'un résultat clé pour les conditions nécessaires sous forme Pontryaguine du premier et du second ordre, valable également pour un minimum de Pontryaguine. La question de la qualification du problème partiellement relaxé est simplifiée dans la thèse par l'introduction d'une variable d'écart θ.

Il suffit ensuite d'appliquer la théorie de l'optimisation dans des espaces de Banach et d'écrire les conditions nécessaires sous forme Lagrange pour ce problème. Ses contraintes supplémentaires sur les contrôles α^i ont des variables duales associées γ^i et la condition de stationnarité de son Hamiltonien augmenté par rapport à la variable α^i donne, pour $1 \leq i \leq N$ et pour presque tout t,

$$H[p^\lambda(t)](t, u^i(t), \bar{y}(t)) - H[p^\lambda(t)](t, \bar{u}(t), \bar{y}(t)) = -\gamma^i(t) \geq 0.$$

Lorsque $N \to \infty$, on obtient l'existence de $\lambda \in \Lambda_P$ si les u^i sont bien choisis ; voir les Lemmes 1.3.4 et 1.3.5. Ce qui prouve le Théorème 2.

La nouveauté avec le Chapitre 1 de la thèse consiste à suivre la même idée pour les conditions nécessaires du second ordre : on écrit ces conditions sous forme Lagrange pour les problèmes partiellement relaxés, et à la limite on obtient des conditions sous forme Pontryaguine. Toutefois, afin de réduire l'écart entre les conditions nécessaires et celles suffisantes du Chapitre 2, on appliquera la relaxation partielle à une variante du problème de contrôle optimal original. En effet, les conditions suffisantes du second ordre font intervenir le *cône critique* C_2, qui est en $(\bar{u}, \bar{y})$ l'ensemble des directions

$$(v, z) \in L^2(0,T;\mathbb{R}^m) \times W^{1,2}(0,T;\mathbb{R}^n)$$

tangentes

– pour la dynamique linéarisée, autrement dit où z est un état linéarisé associé à v :

$$\dot{z}(t) = D_{(u,y)}f(t, \bar{u}(t), \bar{y}(t))(v(t), z(t)) \quad t \in (0,T),$$

– pour les contraintes linéarisées :

$$\begin{aligned} D\Phi(\bar{y}(0), \bar{y}(T))(z(0), z(T)) &\in T_K\left(\Phi(\bar{y}(0), \bar{y}(T))\right), \\ D_{(u,y)}c(t, \bar{u}(t), \bar{y}(t))(v(t), z(t)) &\leq 0 \quad t \in \{c(\cdot, \bar{u}(\cdot), \bar{y}(\cdot)) = 0\}, \\ D_y g(t, \bar{y}(t))z(t) &\leq 0 \quad t \in \{g(\cdot, \bar{y}(\cdot)) = 0\}, \end{aligned}$$

et critiques : $D\phi(\bar{y}(0), \bar{y}(T))(z(0), z(T)) \leq 0$. Or pour ne pas calculer l'effet enveloppe-like de Kawasaki qui apparaît dans les conditions nécessaires du second ordre [34, 58], on peut établir celles-ci sur le sous-cône des directions $(v, z) \in C_2$ *critiques strictes et radiales*, ce que l'on définit par

$$\begin{aligned} D_{(u,y)}c(t, \bar{u}(t), \bar{y}(t))(v(t), z(t)) &= 0, \\ D_y g(t, \bar{y}(t))z(t) &= 0, \end{aligned} \tag{2}$$

pour t dans un voisinage des ensembles de contact

$$\{c(\cdot, \bar{u}(\cdot), \bar{y}(\cdot)) = 0\} \quad \text{et} \quad \{g(\cdot, \bar{y}(\cdot)) = 0\}, \tag{3}$$

respectivement ; voir l'Appendice 1.A.1 à propos du caractère L^2 des directions, qui est un apport du Chapitre 1. En cas de points de contact isolés – que l'on appelle des *touch points* – pour les contraintes pures, ce sous-cône critique strict et radial est beaucoup plus petit que C_2 dans le sens où son adhérence est strictement incluse dans le cône critique strict, défini comme l'ensemble des directions $(v, z) \in C_2$ pour lesquelles les égalités (2) sont satisfaites pour t dans les ensembles de contact (3).

On procède alors par réduction : on fixe une trajectoire admissible $(\bar{u}, \bar{y})$ et au voisinage de chaque point de contact isolé, on reformule la contrainte sur l'état par une contrainte scalaire. Supposons pour simplifier que la contrainte pure sur l'état g ait un unique point de contact isolé $\tau \in (0, T)$; on appelle *problème réduit* le problème de contrôle optimal obtenu en remplaçant la contrainte

$$g(t, y(t)) \leq 0 \quad t \in (0, T)$$

par

$$\begin{cases} \sup\limits_{t \in (\tau - \varepsilon, \tau + \varepsilon)} g(t, y(t)) \leq 0 \\ g(t, y(t)) \leq 0 \quad t \in (0, T) \setminus (\tau - \varepsilon, \tau + \varepsilon) \end{cases}$$

avec $\varepsilon > 0$ fixé. La trajectoire $(\bar{u}, \bar{y})$ est un minimum local du problème initial si et seulement si elle en est un du problème réduit, les multiplicateurs de Lagrange sont inchangés et le cône critique de ce problème d'optimisation auxiliaire est encore C_2 ; son *cône critique strict* C_2^S est l'ensemble des directions $(v, z) \in C_2$ telles que

$$\begin{aligned} D_{(u,y)} c(t, \bar{u}(t), \bar{y}(t))(v(t), z(t)) = 0 \quad & t \in \{c(\cdot, \bar{u}(\cdot), \bar{y}(\cdot)) = 0\}, \\ D_y g(t, \bar{y}(t)) z(t) = 0 \quad & t \in \{g(\cdot, \bar{y}(\cdot)) = 0\} \setminus \{\tau\}, \end{aligned} \tag{4}$$

et le *cône critique strict et radial* C_2^R est défini en considérant (4) sur un voisinage des ensembles de contact, privé de τ pour la contrainte pure. On a maintenant, sous certaines hypothèses sur la structure des ensembles de contact et sur les dérivées des contraintes mixtes et pures le long des trajectoires, la Proposition 1.4.19 :

Proposition 4. *Le cône critique strict et radial C_2^R du problème réduit est dense dans le cône critique strict C_2^S.*

On peut donc établir des conditions nécessaires du second ordre sur C_2^S par densité, simplement en calculant le *Hessien du Lagrangien* du problème réduit ; voir le Théorème 1.A.5. Ce calcul peut être effectué sous l'hypothèse que le point de contact isolé τ est *réductible* :

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\tau, \bar{y}(\tau)) < 0, \tag{5}$$

cette dérivée devant être localement bien définie et continue. On trouve alors, pour tout $\lambda \in \Lambda_L$, la forme quadratique

$$\begin{aligned} \Omega[\lambda](v, z) := \int_0^T D^2_{(u,y)} H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)(v_t, z_t)^2 \mathrm{d}t + D^2 \Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T)^2 \\ + \int_{[0,T]} \mathrm{d}\mu_t D^2 g(t, \bar{y}_t)(z_t)^2 - \mu(\{\tau\}) \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t} D_y g(\tau, \bar{y}(\tau)) z(\tau)\right)^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2} g(\tau, \bar{y}(\tau))}, \end{aligned}$$

où on note la dépendance de t en indice lorsqu'il n'y a pas d'ambiguïté. La même contribution de τ – et plus généralement d'un nombre fini de points de contact isolés – apparaît avec le calcul de l'effet enveloppe-like de Kawasaki dans l'établissement des conditions nécessaires directement sur C_2^S, comme dans [20, 22] sous des hypothèses plus contraignantes.

Les conditions du second ordre sous forme Pontryaguine sont ensuite obtenues en combinant la relaxation partielle à la réduction pour les conditions nécessaires, tandis que les conditions suffisantes reposent sur un principe de décomposition appliqué lui aussi au probléme réduit : le Théorème 2.4.2. Ce résultat majeur du Chapitre 2 généralise le principe de décomposition de Bonnans et Osmolovskii [24] à des problèmes de contrôle optimal avec contraintes mixtes et pures. Il fournit un développement au second ordre du Lagrangien par rapport aux variations de la variable d'optimisation u, dont certaines sont grandes en norme $\|\cdot\|_\infty$ lorsque l'on considère des minima forts. Les deux résultats principaux de la Partie I de la thèse sont alors le Théorème 1.4.9 pour le Chapitre 1 et le Théorème 2.5.3 pour le Chapitre 2, respectivement :

Théorème 5. *Soit $(\bar{u}, \bar{y})$ un minimum borné fort. On suppose satisfaites*
- *la condition rentrante* (1)*,*
- *les hypothèses garantissant la Proposition 4,*
- *l'hypothèse de réductibilité* (5) *du point de contact isolé τ.*

Alors pour tout $(v, z) \in C_2^S$, il existe $\lambda \in \Lambda_P$ tel que

$$\Omega[\lambda](v, z) \geq 0.$$

Théorème 6. *Soit $(\bar{u}, \bar{y})$ une trajectoire admissible. On suppose satisfaites*
- *la condition rentrante* (1)*,*
- *l'hypothèse de réductibilité* (5) *du point de contact isolé τ,*
- *une propriété de régularité métrique de la contrainte mixte,*
- *une propriété de coercivité des formes quadratiques $\Omega[\lambda]$ pour tout $\lambda \in \Lambda_P$.*

S'il existe $\alpha > 0$ et $\lambda^* \in \Lambda_P$ pour lequel le Hamiltonien satisfait, pour presque tout t, la condition de croissance quadratique

$$H[p^{\lambda^*}(t)](t, u, \bar{y}(t)) - H[p^{\lambda^*}(t)](t, \bar{u}(t), \bar{y}(t)) \geq \alpha |u - \bar{u}(t)|^2$$

pour tout u dans l'adhérence de $\{c(t, \cdot, \bar{y}(t)) < 0\}$ et si pour tout $(v, z) \in C_2 \setminus \{0\}$, il existe $\lambda \in \Lambda_P$ tel que

$$\Omega[\lambda](v, z) > 0,$$

alors $(\bar{u}, \bar{y})$ est un minimum borné fort avec croissance quadratique.

Ces conditions du second ordre sous forme Pontryaguine sont obtenues dans la thèse pour des contraintes mixtes et pures vectorielles et avec un nombre fini de points de contact isolés pour chacune des contraintes pures sur l'état ; les conditions nécessaires fournies par le Théorème 5 y sont présentées pour un minimum de Pontryaguine, et les conditions suffisantes du Théorème 6 dans une version légèrement plus faible. La nouveauté de ces résultats réside, par rapport à ceux d'Osmolovskii [69, 73, 71, 72], dans la variété des contraintes – en particulier pures sur l'état – imposées, ou par rapport à ceux de Bonnans et Hermant [20, 22], dans la formulation en terme de multiplicateurs – non uniques – de Pontryaguine et de minima forts. On déduit – au Chapitre 2 – des deux théorèmes précédents le Théorème 2.6.3 qui caractérise la croissance quadratique :

Théorème 7. *Soit $(\bar{u}, \bar{y})$ une trajectoire admissible. On suppose satisfaites*

– *les hypothèses des Théorèmes 5 et 6,*
– *la condition de stricte complémentarité* $C_2^S = C_2$*,*
– *l'hypothèse de bornitude, uniformément en* t*, de* $\{c(t,\cdot,\bar{y}(t)) < 0\}$*,*
– *la condition de non dégénérescence* $\beta > 0$ *pour tout* $\lambda = (\beta, \Psi, \nu, \mu) \in \Lambda_P$*.*

Alors $(\bar{u}, \bar{y})$ *est un minimum borné fort avec croissance quadratique si et seulement si les conditions suffisantes du Théorème 6 sont satisfaites.*

L'hypothèse de bornitude est en fait inutile lorsque l'on considère la version plus faible des conditions suffisantes de la thèse. Une condition nécessaire pour la stricte complémentarité y est donnée, et un résultat intéressant au Chapitre 1 est le Théorème 1.A.14 qui fournit une condition nécessaire et suffisante pour la non dégénérescence des multiplicateurs de Pontryaguine généralisés.

Conditions du second ordre pour des équations intégrales

La Partie II de la thèse commence avec le Chapitre 3 et des conditions d'optimalité pour un problème de la forme (P_C) avec équation d'état intégrale et contraintes terminales et pures sur l'état ; on considère ainsi le problème

$$\begin{aligned} &\min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \phi(y(0), y(T)) \quad \text{sous les contraintes} \\ &y(t) = y(0) + \int_0^t f\,(t, u(s), y(s))\,\mathrm{d}s \quad t \in (0,T), \\ &\Phi(y(0), y(T)) \in K, \\ &g(y(t)) \leq 0 \quad t \in (0,T), \end{aligned} \tag{P_3}$$

où $\mathcal{U} := L^\infty(0,T;\mathbb{R}^m)$, $\mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n)$. Le problème du Chapitre 3 est formulé comme un problème de Bolza et avec une dynamique f qui dépend en plus de la variable d'intégration s, ce qui ne change pas la nature des conditions d'optimalité. Les données y sont supposées de classe C^∞ ; on suppose de plus ici que g est scalaire, afin de simplifier la présentation des résultats.

Les conditions d'optimalité établies au Chapitre 3 le sont pour des minima faibles et qualifiés ; elles font donc intervenir les multiplicateurs de Lagrange, non pas généralisés mais normaux. D'autre part, l'absence de contrainte mixte dans le problème de contrôle optimal considéré rend caduque l'introduction d'une variable duale associée et d'un Hamiltonien augmenté. Comme expliqué dans l'introduction générale, le Hamiltonien – non local en la variable adjointe p – est défini par

$$H[p](t,u,y) := p(t)f(t,u,y) + \int_t^T p(s)\frac{\partial f}{\partial t}(s,u,y)\mathrm{d}s$$

et l'équation adjointe – en présence d'une contrainte pure – est définie dans l'espace des fonctions à variations bornées par

$$\begin{cases} -\mathrm{d}p(t) = D_y H[p](t, u(t), y(t))\mathrm{d}t + \mathrm{d}\mu(t) g'(y(t)) \\ \quad p(T) = D_{y_T}\Phi[\Psi](y(0), y(T)) \\ -p(0) = D_{y_0}\Phi[\Psi](y(0), y(T)) \end{cases}$$

où μ est une mesure et $\Phi[\Psi](y_0, y_T) := \phi(y_0, y_T) + \Psi\Phi(y_0, y_T)$ est le Lagrangien terminal. Les multiplicateurs de Lagrange en $(\bar{u}, \bar{y})$ sont alors les couples (Ψ, μ) appartenant au cône normal à

$$K \times C([0,T]; \mathbb{R}_-) \text{ en } (\Phi(\bar{y}(0), \bar{y}(T)), g(\bar{y}(\cdot)))$$

et tels qu'il existe un état adjoint associé p pour lequel le Hamiltonien satisfait la condition de stationnarité

$$D_u H[p](t, \bar{u}(t), \bar{y}(t)) = 0$$

pour presque tout t. Les conditions du second ordre reposent, comme dans le cas sans mémoire, sur le problème réduit obtenu pour une trajectoire admissible $(\bar{u}, \bar{y})$ fixée en reformulant la contrainte sur l'état au voisinage de ses points de contact isolés. On suppose à nouveau pour simplifier que g a un unique point de contact isolé $\tau \in (0, T)$; sous la même hypothèse de réductibilité – dérivée seconde de g le long de $\bar{y}$ bien définie et continue au voisinage de τ, strictement négative en τ – on peut calculer le Hessien du Lagrangien du prolème réduit.

La difficulté principale pour établir les conditions nécessaires du second ordre, qui a été éludée au paragraphe sur les conditions sous forme Pontryaguine et qui nécessite un travail spécifique pour les équations intégrales, réside dans la preuve de la densité du cône des directions critiques strictes et radiales dans celui des directions critiques strictes ; c'est un résultat majeur du Chapitre 3. On suppose – encore pour simplifier l'introduction – que la condition initiale $y(0)$ est fixée à y_0, avec $g(y_0) < 0$. La trajectoire admissible $(\bar{u}, \bar{y})$ étant fixée, on note $z[v]$ l'unique état linéarisé associé à une direction v, solution de

$$z(t) = \int_0^t D_{(u,y)} f(t, \bar{u}(s), \bar{y}(s))(v(s), z(s)) \mathrm{d}s \quad t \in (0, T).$$

Soit Δ l'ensemble de contact de la contrainte sur l'état pour le problème réduit, auquel on a donc retiré le point de contact isolé τ :

$$\Delta := \{t \,:\, g(\bar{y}(t)) = 0 \text{ et } t \neq \tau\}.$$

Le *cône critique* C_2 en $(\bar{u}, \bar{y})$ est alors l'ensemble des directions

$$(v, z) \in L^2(0, T; \mathbb{R}^m) \times W^{1,2}(0, T; \mathbb{R}^n)$$

tangentes et critiques, soit $z = z[v]$ et

$$\begin{aligned} D\phi(\bar{y}(0), \bar{y}(T))(0, z(T)) &\leq 0, \\ D\Phi(\bar{y}(0), \bar{y}(T))(0, z(T)) &\in T_K\left(\Phi(\bar{y}(0), \bar{y}(T))\right), \\ g'(\bar{y}(\tau))z(\tau) &\leq 0 \\ g'(\bar{y}(\cdot))z(\cdot) &\leq 0 \text{ sur } \Delta. \end{aligned}$$

Le *cône critique strict* C_2^S est l'ensemble des directions $(v, z) \in C_2$ telles que

$$g'(\bar{y}(\cdot))z(\cdot) = 0 \text{ sur } \Delta$$

et le *cône critique strict et radial* C_2^R est l'ensemble des directions $(v, z) \in C_2$ telles que

$$g'(\bar{y}(\cdot))z(\cdot) = 0 \text{ sur un voisinage de } \Delta.$$

On est intéressé par la densité de C_2^R dans C_2^S. Par un lemme de Dmitruk [38, Lemme 1], il suffit de montrer la densité des espaces vectoriels sous-jacents, sans prendre en compte les contraintes polyédriques qui définissent les cônes. Il s'agit donc de prouver que

$$\left\{v \in L^2(0, T; \mathbb{R}^m) \,:\, g'(\bar{y}(\cdot))z[v](\cdot) = 0 \text{ sur un voisinage de } \Delta\right\}$$

est dense dans

$$\left\{v \in L^2(0, T; \mathbb{R}^m) \,:\, g'(\bar{y}(\cdot))z[v](\cdot) = 0 \text{ sur } \Delta\right\},$$

ce qui passe par deux étapes :

1. la densité de
$$\{h \in X \,:\, h = 0 \text{ sur un voisinage de } \Delta\}$$
dans
$$\{h \in X \,:\, h = 0 \text{ sur } \Delta\}\,.$$
2. la surjectivité de
$$v \in L^2 \mapsto g'(\bar{y}(\cdot))z[v](\cdot) \in X.$$

Se pose alors la question de l'espace X à considérer ; la réponse tient à la régularité de g le long des trajectoires et à la notion d'*ordre* d'une contrainte pure sur l'état. Un apport du Chapitre 3 est la clarification de cette notion, généralisée aux équations intégrales par Bonnans et De la Vega [16], et l'extension de propriétés qui y sont liées ; voir la Section 3.2.4.

Etant donnée une trajectoire (u, y), $g(y(\cdot)) \in W^{1,\infty}(0, T)$ a priori et
$$\frac{\mathrm{d}}{\mathrm{d}t} g(y(t)) = g'(y(t)) \left(f(t, u(t), y(t)) + \int_0^t \frac{\partial f}{\partial t}(t, u(s), y(s))\mathrm{d}s \right).$$

Soit $g^{(1)} \colon \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{Y} \to \mathbb{R}$ définie par
$$g^{(1)}(t, \tilde{u}, \tilde{y}, \hat{u}, \hat{y}) := g'(\tilde{y}) \left(f(t, \tilde{u}, \tilde{y}) + \int_0^t \frac{\partial f}{\partial t}(t, \hat{u}(s), \hat{y}(s))\mathrm{d}s \right),$$
de sorte que pour toute trajectoire (u, y),
$$\frac{\mathrm{d}}{\mathrm{d}t} g(y(t)) = g^{(1)}(t, u(t), y(t), u, y).$$

Si $D_{\tilde{u}} g^{(1)} \equiv 0$, alors en fait $g(y(\cdot)) \in W^{2,\infty}(0, T)$ et on peut définir $g^{(2)}$ de sorte que
$$\frac{\mathrm{d}}{\mathrm{d}t} g^{(1)}(t, y(t), u, y) = g^{(2)}(t, u(t), y(t), u, y).$$

Et ainsi de suite, avec une difficulté par rapport au cas sans mémoire pour définir formellement les fonctions $g^{(j)}$ du fait des termes locaux et non locaux en u qu'il faut distinguer à chaque étape ; c'est fait à la Section 3.2.4. L'ordre d'une contrainte pure [65] se généralise ensuite naturellement avec la Définition 3.2.9 :

Définition 8. L'*ordre* d'une contrainte pure sur l'état g est le plus grand entier q tel que
$$D_{\tilde{u}} g^{(j)} \equiv 0 \quad \text{pour } j < q.$$

Soit q l'ordre de la contrainte g. Par définition, $g(y(\cdot)) \in W^{q,\infty}(0, T)$ et
$$\frac{\mathrm{d}^q}{\mathrm{d}t^q} g(y(t)) = g^{(q)}(t, u(t), y(t), u, y)$$
pour toute trajectoire (u, y). On prouve de plus un principe de commutativité avec la linéarisation, au sens du Lemme 3.2.10 :

Lemme 9. *Pour tout $v \in L^2(0, T; \mathbb{R}^m)$, $g'(y(\cdot))z[v](\cdot) \in W^{q,2}(0, T)$ et en notant $z := z[v]$, on trouve*
$$\frac{\mathrm{d}^q}{\mathrm{d}t^q} g'(\bar{y}(t))z(t) = D_{(\tilde{u},\tilde{y},u,y)} g^{(q)}(t, \bar{u}(t), \bar{y}(t), \bar{u}, \bar{y})(v(t), z(t), v, z).$$

L'espace X qui s'impose dans le schéma de preuve présenté est alors $W^{q,2}(0,T)$; reprenant les deux étapes :

1. la densité est assurée sous l'hypothèse que Δ a un nombre fini de composantes connexes ; voir le Lemme 3.4.10, qui n'est pas spécifique aux équations intégrales, pour un résultat plus fort. Il est ici indispensable d'avoir retiré τ de l'ensemble de contact Δ pour que chacune de ses composantes soit d'intérieur non vide et qu'on ait ainsi
$$h = 0 \text{ sur } \Delta \quad \Rightarrow \quad \frac{\mathrm{d}^j h}{\mathrm{d}t^j} = 0 \text{ sur } \Delta$$
pour $0 \leq j \leq q$; c'est ce qui motive la réduction.
2. la surjectivité est impliquée par la condition d'inversibilité :
$$\left| D_{\tilde{u}} g^{(q)}(t, \bar{u}(t), \bar{y}(t), \bar{u}, \bar{y}) \right| > 0 \tag{6}$$
uniformément en t ; il s'agit de la version simplifiée d'un résultat clé du Chapitre 3, établi pour les équations intégrales par le Lemme 3.4.2. La condition d'inversibilité devient une condition d'indépendance linéaire dans le cas de contraintes sur l'état vectorielles.

En suivant ce schéma, on peut prouver le résultat majeur annoncé qu'est le Lemme 3.4.9 :

Lemme 10. *On suppose que*
- *l'ensemble Δ a un nombre fini de composantes connexes,*
- *la condition d'inversibilité* (6) *de la dérivée ultime de g est satisfaite.*

Alors le cône critique strict et radial C_2^R du problème réduit est dense dans le cône critique strict C_2^S.

La densité est établie au Chapitre 3 pour des contraintes sur l'état vectorielles et pour des directions critiques strictes et radiales dans L^∞. Cette restriction à L^∞ demande plus de travail mais permet d'appliquer simplement les conditions nécessaires du second ordre pour un problème abstrait d'optimisation dans L^∞, au lieu des conditions obtenues directement sur un cône dans L^2 à l'Appendice 1.A.1 du Chapitre 1.

La condition d'inversibilité (6) de la dérivée ultime de g peut déjà être exploitée pour les conditions nécessaires du premier ordre, comme dans le Théorème 3.4.6 :

Théorème 11. *Soit $(\bar{u}, \bar{y})$ un minimum faible qualifié. On suppose que la condition d'inversibilité* (6) *est satisfaite. Alors l'ensemble Λ des multiplicateurs de Lagrange est un convexe de dimension finie, non vide et compact.*

Avec les notations introduites pour définir l'ordre d'une contrainte pure sur l'état, le Hessien du Lagrangien pour le problème réduit peut s'écrire, pour tout $\lambda \in \Lambda$, comme la forme quadratique

$$\begin{aligned} \Omega[\lambda](v) := \int_0^T D^2_{(u,y)} H[p](t, \bar{u}_t, \bar{y}_t)(v_t, z_t)^2 \mathrm{d}t + D^2 \Phi[\Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T)^2 \\ - \mu(\{\tau\}) \frac{\left(D_{(\tilde{u}, \tilde{y}, u, y)} g^{(1)}(\tau, \bar{u}_\tau, \bar{y}_\tau, \bar{u}, \bar{y})(v_\tau, z_\tau, v, z) \right)^2}{g^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau, \bar{u}, \bar{y})} \end{aligned}$$

où $z := z[v]$ et la dépendance en t est notée en indice. Les résultats principaux du Chapitre 3 sont ensuite les Théorèmes 3.4.8 et 3.4.12, respectivement :

Théorème 12. *Soit $(\bar{u}, \bar{y})$ un minimum faible qualifié. On suppose que les hypothèses du Lemme 10 sont satisfaites et que le point de contact isolé τ est réductible. Alors pour tout $(v, z) \in C_2^S$, il existe $\lambda \in \Lambda$ tel que*

$$\Omega[\lambda](v, z) \geq 0.$$

Théorème 13. *Soit $(\bar{u}, \bar{y})$ une trajectoire admissible. On suppose que le point de contact isolé τ est réductible et que les formes quadratiques $\Omega[\lambda]$ satisfont pour tout $\lambda \in \Lambda$ une propriété de coercivité. Si pour tout $(v, z) \in C_2 \setminus \{0\}$, il existe $\lambda \in \Lambda$ tel que*

$$\Omega[\lambda](v, z) > 0,$$

alors $(\bar{u}, \bar{y})$ est un minimum faible avec croissance quadratique.

Ces conditions du second ordre – nécessaires et suffisantes – sont une première pour des problèmes de contrôle optimal d'équations intégrales. Les contraintes pures – vectorielles et avec un nombre fini de points de contact isolé au Chapitre 3 – et terminales sur l'état – conduisant à la non unicité des multiplicateurs – en font une extension des résultats de Bonnans et Hermant [21]. On en déduit une caractérisation de la croissance quadratique pour un minimum faible avec le Corollaire 3.4.14 :

Corollaire 14. *Soit $(\bar{u}, \bar{y})$ une trajectoire admissible qualifiée. On suppose satisfaites*

- *les hypothèses des Théorèmes 12 et 13,*
- *la condition de stricte complémentarité $C_2^S = C_2$.*

Alors $(\bar{u}, \bar{y})$ est un minimum faible avec croissance quadratique si et seulement si pour tout $(v, z) \in C_2 \setminus \{0\}$, il existe $\lambda \in \Lambda$ tel que

$$\Omega[\lambda](v, z) > 0.$$

2.2 Modélisation et dynamique de populations

Version contrôlée du modèle de Mackey

La Partie II de la thèse se poursuit avec le Chapitre 4 et la modélisation d'expériences biologiques où l'on cherche à minimiser la prolifération de cellules cancéreuses, qui fournit un exemple de problème de contrôle optimal d'équations différentielles avec mémoire. Ce travail de modélisation s'appuie sur une collaboration avec les expérimentateurs-biologistes, des médecins et des mathématiciens, regroupés au sein du projet ALMA (Analyse de la Leucémie Myéloblastique Aiguë) ; l'objectif ultime de ce projet est l'optimisation thérapeutique pour ces cancers du sang. Les expériences réalisées consistent à

- prélever du sang de patients atteints d'une leucémie aiguë myéloblastique (LAM)
- isoler les cellules hématopoïétiques cancéreuses immatures, présentes dans le sang du fait de la LAM,
- les mettre en culture pendant 5 jours, en présence ou non de deux médicaments,
- mesurer quotidiennement le nombre de cellules, leur position dans le cycle cellulaire et leur niveau de différentiation ;

un article décrivant précisément la méthode expérimentale est en préparation [9]. On cherche dans un premier temps des protocoles d'administration in vitro des médicaments qui soient le plus efficace possible sans être trop toxiques, ce que l'on formule comme un problème de contrôle optimal.

Les modèles de dynamique de populations de cellules hématopoïètiques sous forme d'EDP structurées en âge, qui justifient les modèles à retard de type Mackey [1, 62], permettent de représenter l'action des deux médicaments. Ces deux médicaments ont des actions différentes sur le cycle cellulaire : l'un – représenté par u – augmente la mort cellulaire, l'autre – représenté par k – ralentit la prolifération. En intégrant le long des caractéristiques, on obtient un système d'équations différentielles, sans mémoire pour des temps t proches de l'instant initial 0, puis avec mémoire – un retard discret, puis deux retards discrets et un retard distribué – lorsque t augmente. Ainsi, pour $t > 2\tau$, on reconnaît une version contrôlée du modèle de Mackey [62] :

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = -(1-k(t))\beta R(t) + 2(1-k(t-2\tau)\beta R(t-2\tau)e^{-\left(\gamma 2\tau + \int_{t-\tau}^{t} u(s)\mathrm{d}s\right)} \tag{7}$$

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t}(t) &= -(\gamma P(t) + u(t)P_2(t)) + (1-k(t))\beta R(t) \\ &\quad - (1-k(t-2\tau))\beta R(t-2\tau)e^{-\left(\gamma 2\tau + \int_{t-\tau}^{t} u(s)\mathrm{d}s\right)}\end{aligned} \tag{8}$$

$$\frac{\mathrm{d}P_2}{\mathrm{d}t}(t) = -(\gamma + u(t))P_2(t) + (1-k(t-\tau))\beta R(t-\tau)e^{-\gamma\tau} \tag{9}$$

où les variables d'état R, P, P_2 sont des sous-populations de cellules. Dans le Chapitre 4, k est également une variable d'état, dont la dynamique dépend d'un contrôle v.

Le problème de contrôle optimal que l'on considère est alors le suivant : minimiser la prolifération, pour une dynamique de populations donnée par (7)-(9) et sous des contraintes sur les doses cumulées de médicaments. L'existence d'un protocole d'administration optimal pour ce problème tient au fait que la dynamique est affine en les contrôles ; c'est l'objet de la Proposition 4.5.1 :

Proposition 15. *Il existe au moins un protocole d'administration optimal.*

En vue de caractériser ce ou ces protocoles optimaux, on énonce le principe de Pontryaguine pour le problème considéré – de contrôle optimal d'équations différentielles avec mémoire – avec le Théorème 4.A.1. La synthèse des protocoles optimaux n'y est pas obtenue ; aux difficultés habituelles s'ajoute le fait que la dynamique adjointe est donnée par des équations différentielles à arguments avancés. On peut malgré tout tirer certaines informations du principe de Pontryaguine, par exemple la Proposition 4.5.5 :

Proposition 16. *Si la première valeur propre du système non contrôlé est positive et si la dose cumulée du médicament qui ralentit la prolifération n'est pas contrainte, alors il est optimal d'en administrer à la fin de l'expérience.*

La transformation – dite de Guinn – du problème avec retards en un problème sans retard [48, 49] permet l'utilisation de boîtes à outils pour le contrôle optimal d'EDO pour le résoudre numériquement. Certains protocoles optimaux ainsi obtenus sont présentés dans le Chapitre 4.

Entropie relative généralisée

La fonction objectif – et les contraintes – du problème de contrôle optimal précédent est bien sûr explicitée au Chapitre 4. L'approche naïve consistant à minimiser la population totale $R+P$ à l'instant final T de l'expérience se heurte à un effet d'horizon. La définition d'une fonction coût convenable nécessite, suivant une idée de Thomas Lepoutre [1], d'étudier

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le comportement en temps long du modèle structuré en âge, sans l'action des médicaments :

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = -\beta R(t) + 2p(t,2\tau) \qquad 0 < t, \tag{10}$$
$$\frac{\partial p}{\partial t}(t,a) + \frac{\partial p}{\partial a}(t,a) = -\gamma p(t,a) \qquad 0 < t,\ 0 < a < 2\tau, \tag{11}$$
$$p(t,0) = \beta R(t) \qquad 0 < t. \tag{12}$$

Cette étude asymptotique est basée sur le principe d'entropie relative généralisée introduit par Michel, Mischler et Perthame [68]. Un résultat important du Chapitre 4 est le Théorème 4.3.5, qui établit une version précise de ce principe pour le modèle (10)-(12). L'entropie relative généralisée permet de comparer une solution quelconque du modèle structuré à des solutions particulières formées à partir des premiers éléments propres, dont l'existence est fournie par le Théorème 4.3.7 :

Théorème 17. *Il existe un unique quintuplet* $(\lambda, \bar{R}, \bar{p}, \bar{\Psi}, \bar{\phi})$ *de premiers éléments propres.*

Les solutions particulières en question sont les $\rho(\bar{R}e^{\lambda t}, \bar{p}(\cdot)e^{\lambda t})$, $\rho \in \mathbb{R}$; on montre, en adaptant les preuves de [68, 77], que toute solution (R,p) converge en temps long vers une solution particulière, au sens du Théorème 4.3.8 :

Théorème 18. *Soit* (R,p) *une solution de* (10)-(12). *Il existe* $\rho \in \mathbb{R}$ *tel que*

$$\lim_{t\to\infty} \left(\bar{\Psi}|R(t)e^{-\lambda t} - \rho\bar{R}| + \int_0^{2\tau} \bar{\phi}(a)|p(t,a)e^{-\lambda t} - \rho\bar{p}(a)|\mathrm{d}a \right) = 0. \tag{13}$$

Ce résultat est fondamental pour le Chapitre 4 puisqu'il justifie que l'on définisse la fonction objectif du problème de contrôle optimal comme étant ce poids ρ, à minimiser à l'issue d'une phase $[0,T]$ où les médicaments agissent. Cette définition se distingue de celles existantes dans la littérature sur l'optimisation thérapeutique pour les cancers, telles que le nombre final ou maximal de cellules d'une tumeur [10], son volume final [59], un indice de performance [60] ou la valeur propre d'un problème périodique [14] ; elle constitue une nouvelle approche.

3 Perspectives

3.1 Conditions d'optimalité pour des problèmes avec mémoire

Il est possible d'étendre les conditions d'optimalité sous forme Pontryaguine des Chapitres 1 et 2 à certains problèmes de la forme (P_C) avec mémoire.

Une première étape, qui possède un intérêt en soi, est d'obtenir des conditions d'optimalité sous forme Lagrange pour un minimum faible. Il s'agit donc de généraliser les résultats du Chapitre 3 à d'autres familles de mesures $(\nu_t)_t$;

- au premier ordre, cela passe par le Hamiltonien non local défini dans l'introduction générale et peut se faire sous réserve que la seconde marginale ν de $\nu_t \otimes \mathcal{L}^1$ soit suffisamment régulière.
- au second ordre, en l'absence de contraintes pures sur l'état, rien de plus ne s'oppose à la généralisation.
- au second ordre, en présence de contraintes pures sur l'état, il faut pouvoir définir la notion d'ordre d'une contrainte pour la dynamique considérée et prouver la densité du cône critique strict et radial pour le problème réduit – la réduction ne présentant pas de difficulté – dans le cône critique strict, ce qui doit être fait au cas par cas.

Des conditions du premier ordre sont obtenues par Carlier et Tahraoui [27] ; il n'en n'existe pas de connues au second ordre en général pour des problèmes de contrôle optimal d'équations différentielles avec mémoire. Elles seront utilisées dans un travail à venir en analyse numérique.

La seconde étape, pour les conditions nécessaires, consiste à appliquer la relaxation partielle à des problèmes avec mémoire. Le cœur de la relaxation est la Proposition 1.3.8, qui permet d'obtenir un minimum faible du problème relaxé à partir d'un minimum fort du problème original, dit classique. Cette proposition est un résultat d'approximation des trajectoires relaxées par des trajectoires classiques ; on parle parfois de théorèmes de relaxation [37]. Sa preuve dans l'Appendice 1.A.2 du Chapitre 1 repose sur un théorème de Liapounov [52, 61] que l'on peut formuler de la façon suivante :

Théorème. *Soient* $f^1, \dots, f^N \in L^1(0,1;\mathbb{R}^K)$ *. Pour tous* $\alpha^1, \dots, \alpha^N \in L^\infty(0,1)$ *tels que, pour presque tout* t*,* $\alpha^i(t) \in [0,1]$ *et* $\sum \alpha_i(t) = 1$*, il existe* $\hat{\alpha}^1, \dots, \hat{\alpha}^N \in L^\infty(0,1)$ *tels que, pour presque tout* t*,* $\hat{\alpha}^i(t) \in \{0,1\}$*,* $\sum \alpha_i(t) = 1$ *et*

$$\int_0^1 \left(\sum_{i=1}^N \alpha^i(t) f^i(t) \right) \mathrm{d}t = \int_0^1 \left(\sum_{i=1}^N \hat{\alpha}^i(t) f^i(t) \right) \mathrm{d}t.$$

En appliquant ce théorème aux fonctions – essentiellement bornées dans le cadre qui nous intéresse – définies par

$$f^i(t) := \left\langle f\left(t, u^i(\cdot), y(\cdot)\right), \nu_t \right\rangle,$$

on peut prouver un théorème de relaxation pour la dynamique avec mémoire correspondante, dont on déduit des conditions nécessaires sous forme Pontryaguine comme au Chapitre 1. Au premier ordre, on obtiendrait ainsi le principe du minimum de Pontryaguine pour des problèmes de contrôle optimal d'équations différentielles avec mémoire, dans un cadre proche de celui de Halanay [50] mais incluant les contraintes – en dimension infinie – mixtes sur le contrôle et l'état et pures sur l'état. De telles contraintes sont bien considérées dans [6], mais la mémoire ne porte pas sur le contrôle. D'autre part, l'approche par relaxation partielle est plus élémentaire que celle de relaxation par mesures de Young utilisée par Warga [91] ou Rosenblueth et Vinter [82], qui ne considèrent de toute façon pas de contraintes aussi générales.

Pour les conditions suffisantes, il faut vérifier que le principe de décomposition du Chapitre 2 s'étend aux équations différentielles considérées, ce qui sera le cas après avoir établi une version du lemme de Gronwall dont on a également besoin pour les conditions nécessaires.

3.2 Analyse numérique d'un problème avec retard distribué

Un travail en cours concerne l'analyse numérique d'un problème de la forme (P_C) avec équation d'état à retard distribué et contraintes finales sur l'état. Pour le problème

$$\begin{aligned}
&\min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \phi(y(T)) \quad \text{sous les contraintes} \\
&\dot{y}(t) = \int_{t-\tau}^{t} f\left(t, u(s), y(s)\right) \qquad t \in (\tau, T), \\
&u(t) = u^0(t), \quad y(t) = y^0(t) \quad t \in (0,\tau), \\
&\Phi(y(T)) = 0,
\end{aligned} \tag{P}$$

où $\mathcal{U} := L^\infty(0,T;\mathbb{R}^m)$, $\mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n)$, il s'agit de montrer que sous certaines conditions, les méthodes directes fournissent une approximation d'une trajectoire localement optimale. Les métodes directes consistent à discrétiser d'abord et optimiser ensuite [87]. Pour la première étape, on fixe un pas $h = T/N$ et on discrétise l'équation d'état avec le schéma d'Euler explicite où l'on approxime l'intégrale par la méthode des rectangles à gauche. On considère ensuite le problème d'optimisation en dimension finie suivant :

$$\begin{aligned}
&\min_{(u_h,y_h)\in\mathbb{R}^N\times\mathbb{R}^{N+1}} \phi(y_N) \quad \text{sous les contraintes}\\
&y_{n+1} = y_n + h\sum_{l=1}^{j} hf(t_n, u_{n-l}, y_{n-l}) \quad j \le n \le N-1,\\
&u_n = u^0(t_n) \quad 0 \le n \le j-1,\\
&y_n = y^0(t_n) \quad 0 \le n \le j,\\
&\Phi(y_N) = 0,
\end{aligned} \tag{P_h}$$

où $j := [\tau/h]$. Il s'agit d'un problème de programmation non linéaire avec contraintes d'égalité, qui peut être résolu numériquement par une méthode de Newton standard. Le cas avec contraintes d'inégalité sur l'état final pourrait être traité par exemple par programmation quadratique séquentielle ou par une méthode de points intérieurs [12].

La question à laquelle on s'intéresse est de garantir l'existence d'une solution $(\hat{u}_h, \hat{y}_h)$ du problème discrétisé (P_h) – pour tout h suffisamment petit – qui converge vers une solution $(\bar{u}, \bar{y})$ du problème continue lorsque $h \to 0$. On fixe donc une trajectoire $(\bar{u}, \bar{y})$ localement optimale pour le problème (P) et on suppose qu'elle est qualifiée, que $\bar{u}$ est Lipschitz et que les conditions suffisantes du second ordre – fournies par le paragraphe précédent – sont satisfaites. On procède ensuite par homotopie : à h fixé, on construit une famille de problèmes d'optimisation discrets (P_h^θ) telle que $(P_h^1) = (P_h)$ et telle qu'on connaisse une solution de (P_h^0). Par une analyse de sensibilité rendue possible grâce aux conditions suffisantes du second ordre [25], on obtient l'existence d'une solution de (P_h^1) qui répond à la question.

Cette question d'analyse numérique est traitée par Dontchev et Hager dans [39], pour un problème de contrôle optimal d'équations différentielles sans mémoire mais avec des contraintes pures sur l'état ; leur preuve ne repose pas sur une méthode d'homotopie mais sur un résulat abstrait de stabilité. La méthode de Guinn [49] pour se ramener à un problème sans retard – en dimension finie – ne s'adapte pas au retard distribué, ce qui rend impossible d'appliquer l'argument de Göllmann *et al.* [48].

Des généralisations de ce travail à des problèmes avec d'autres contraintes ou à de meilleurs schémas de discrétisation que celui d'Euler explicite sont envisageables. On rappelle que pour des problèmes avec mémoire, la dynamique adjointe est à argument avancé et ne peut donc pas s'intégrer à partir d'une valeur initiale, ce qui empêche la mise en place de méthodes de tir [15].

3.3 Application médicale

Le Chapitre 4 offre de nombreuses perspectives. Parmi les résultats théoriques, le comportement en temps long a été établi pour le modèle structuré en âge (10)-(12) à coefficients constants avec le Théorème 4.3.8 : toute solution (R,p) est telle que

$$(R(t), p(t,\cdot)) \sim \rho\,(\bar{R}e^{\lambda t}, \bar{p}(\cdot)e^{\lambda t})$$

pour la topologie L^1 et lorsque $t \to \infty$, c'est-à-dire au sens de (13) ; ρ est déterminé explicitement en fonction des conditions initiales et des éléments propres. Dans le problème d'optimisation thérapeutique que l'on modélise, les coefficients ne sont en fait pas tous constants lorsqu'on arrête d'administrer les médicaments : β est de la forme

$$\beta(t) = \left(1 - k_0 e^{-\alpha t}\right) \beta.$$

Il semble que cette perturbation de β ne modifie pas radicalement le comportement asymptotique des solutions du modèle structuré correspondant. Des résultats partiels ont été obtenu dans ce sens [2] avec le Lemme 4.3.10 : le paramètre de Malthus est encore donné par la première valeur propre λ et il existe ρ' que l'on imagine être le poids asymptotique par rapport aux mêmes solutions particulières. On espère prouver un résultat de convergence pour la topologie L^2 :

Conjecture 19. *Soit* (R, p) *une solution de* (10)-(12) *avec* $\beta(t) = (1 - k_0 e^{-\alpha t})\,\beta$. *Alors*

$$\lim_{t\to\infty} \left(\frac{\bar{\Psi}}{\bar{R}} \left| R(t) e^{-\lambda t} - \rho' \bar{R} \right|^2 + \int_0^{2\tau} \frac{\bar{\phi}(a)}{\bar{p}(a)} \left| p(t,a) e^{-\lambda t} - \rho' \bar{p}(a) \right|^2 \mathrm{d}a \right) = 0.$$

En effet, une piste pour y arriver est de montrer que pour β constant il y a décroissance exponentielle vers les solutions particulières [45], ce qui semble être exclu pour la topologie L^1 mais possible pour la topologie L^2 :

Conjecture 20. *Soit* (R, p) *une solution de* (10)-(12) *avec* β *constant. Alors*

$$\left(\frac{\bar{\Psi}}{\bar{R}} \left| R(t) e^{-\lambda t} - \rho \bar{R} \right|^2 + \int_0^{2\tau} \frac{\bar{\phi}(a)}{\bar{p}(a)} \left| p(t,a) e^{-\lambda t} - \rho \bar{p}(a) \right|^2 \mathrm{d}a \right) \le C e^{-\mu t},$$

avec C *et* $\mu > 0$.

Des résultats de décroissance exponentielle sont obtenus pour la topologie L^2 dans [26] ; ils reposent à nouveau sur le principe d'entropie relative généralisée, qui a été établi pour le modèle à coefficients variables du Chapitre 4 au Théorème 4.3.5.

Un résultat intéressant serait bien sûr de déterminer explicitement ρ', donné par la limite I_∞ au Lemme 4.3.10 ; cela fournirait peut-être une autre fonction objectif pour le problème de contrôle optimal. Notons qu'ici, avec une seule phase d'administration des médicaments, on ne peut pas modifier le taux de croissance à long terme λ. Une approche complètement différente pour ce problème d'optimisation thérapeutique pourrait être la suivante :

– on considère le modèle structuré en âge (10)-(12) avec des coefficients périodiques, modélisant l'action d'un protocole d'administration répété périodiquement.
– on montre l'existence d'une valeur propre, dite de Floquet par opposition à celle de Perron [32] obtenue dans le cas à coefficients constants avec le Théorème 4.3.7.
– on cherche à minimiser cette valeur propre, qui caractérise la croissance exponentielle pour le modèle périodique [68].

Il s'agit de l'approche développée par Billy *et al.* dans [14] pour des problèmes de chronothérapie.

Du point de vue de l'application médicale à propement parler et du projet ALMA, les paramètres sont en cours d'estimation à partir des données expérimentales pour un modèle plus élaboré que celui du Chapitre 4, sans l'action des médicaments [9]. Il sera peut-être

2. suite à une seconde visite à Lyon

nécessaire de les estimer directement sur notre modèle simplifié à retards avant de résoudre numériquement le problème de contrôle optimal correspondant, le tout avec BOCOP [23] et après transformation de Guinn [49] ; on pourra ensuite suggérer des protocoles in vitro aux biologistes. D'autre part, à défaut d'obtenir la synthèse optimale théorique des protocoles d'administration, il pourrait être intéressant de pousser l'analyse qualitative – protocoles bang-bang, arcs singuliers – comme dans les travaux de Ledzewicz *et al.* [59, 60] ou ceux de Clément *et al.* [33]. Enfin, le modèle retenu au Chapitre 4 peut être développé en distinguant plusieurs niveau de maturité au sein de la population de cellules [2, 7, 74], en couplant des polulations de cellules saines et leucémiques ainsi struturées [86], en prenant en compte la pharmacocinétique-pharmacodynamique (PK-PD) de l'administration des médicaments in vivo [10, 59, 60] ; ces développements rendent hors de portée immédiate l'identification de paramètres mais présentent un intérêt théorique.

Part I

Optimal control of differential equations without memory

Chapter 1

Necessary conditions in Pontryagin form

This chapter is taken from [18]:

J.F. Bonnans, X. Dupuis, L. Pfeiffer. *Second-order necessary conditions in Pontryagin form for optimal control problems.* Submitted. Inria Research Report No. 8306, May 2013.

In this paper, we state and prove first- and second-order necessary conditions in Pontryagin form for optimal control problems with pure state and mixed control-state constraints. We say that a Lagrange multiplier of an optimal control problem is a Pontryagin multiplier if it is such that Pontryagin's minimum principle holds, and we call optimality conditions in Pontryagin form those which only involve Pontryagin multipliers. Our conditions rely on a technique of partial relaxation, and apply to Pontryagin local minima.

1.1 Introduction

The optimization theory in Banach spaces, in particular optimality conditions of order one [79, 92] and two [34, 58, 66], applies to optimal control problems. With this approach, constraints of various kind can be considered, and optimality conditions are derived for weak local minima of optimal control problems. Second-order necessary and sufficient conditions are thereby obtained by Stefani and Zezza [85] in the case of mixed control-state equality constraints, or by Bonnans and Hermant [22] in the case of pure state and mixed control-state constraints. These optimality conditions always involve Lagrange multipliers.

Another class of optimality conditions, necessary and of order one, for optimal control problems comes from Pontryagin's minimum principle. Formulated in the historical book [78] for basic problems, including first-order pure state constraints, this principle has then been extended by many authors. Mixed control-state constraints enter for example the framework developed by Hestenes [54], whereas pure state, and later pure state and mixed control-state, constraints are treated in early Russian references such as the works of Milyutin and Dubovitskii [40, 41], as highlighted by Dmitruk [36]. Let us mention the survey by Hartl et al. [51] and its bibliography for more references on Pontryagin's principles.

Second-order optimality conditions are said in this article to be in Pontryagin form if they only involve Lagrange multipliers for which Pontryagin's minimum principle holds. This restriction to a subset of multipliers is a challenge for necessary conditions, and enables sufficient conditions to give strong local minima. To our knowledge, such conditions have been stated for the first time, under the name of quadratic conditions, for problems with mixed control-state equality constraints by Milyutin and Osmolovskii [69]. Proofs are given by Osmolovskii and Maurer [73], under a restrictive full-rank condition for the mixed equality constraints, that could not for instance be satisfied by pure state constraints.

The main novelty of this paper is to provide second-order necessary conditions in Pontryagin form for optimal control problems with pure state and mixed control-state constraints. We use the same technique as Dmitruk in his derivation of Pontryagin's principle for a general optimal control problem [36]: a partial relaxation of the problem, based on the sliding modes introduced by Gamkrelidze [47]. These convexifications of the set of admissible velocities furnish a sequence of auxiliary optimal control problems, and at the limit, necessary conditions appear to be in Pontryagin form. We thereby get our own version of Pontryagin's minimum principle, as first-order necessary conditions. Then, combining the partial relaxation with a reduction approach [20, 55] and a density argument [17], we obtain second-order necessary conditions in Pontryagin form for a Pontryagin local minimum of our problem. This technique requires to consider a variant of the previous auxiliary problems, but not to compute any envelope-like effect of Kawasaki [58]. Another result that is worth being mentioned is the second-order necessary conditions for a local solution of an abstract optimization problem, that we apply to the partially relaxed problems. We derive them directly on a large set of directions in L^2, which then simplifies the density argument, compared with [17], and avoid a flaw that we will mention in the proof of the density result in [22].

Second-order sufficient conditions for strong local minima of similar optimal control problems constitute another work by the same authors [19]. They rely on an extension of the decomposition principle of Bonnans and Osmolovskii [24], and on the reduction approach. Quadratic growth for a strong local minimum is then characterized.

The paper is organized as follows. In Section 1.2, we set our optimal control problem and define various notions of multipliers and of minima. Section 1.3 is devoted to the first-order necessary conditions: they are stated, in the form of Pontryagin's minimum principle, in Section 1.3.1; our partial relaxation approach is detailed in Section 1.3.2, and then used to prove the first-order conditions in Section 1.3.3. Section 1.4 is devoted to the second-order necessary conditions: they are stated in Section 1.4.1, and proved in Section 1.4.2 by partial relaxation combined with reduction and density. We have postponed our abstract optimization results to Appendix 1.A.1, the proof of an approximation result needed for the partial relaxation to Appendix 1.A.2, a qualification condition to Appendix 1.A.3, and an example about Pontryagin's principle to Appendix 1.A.4.

Notations For a function h that depends only on time t, we denote by h_t its value at time t, by $h_{i,t}$ the value of its ith component if h is vector-valued, and by $\dot{h}$ its derivative. For a function h that depends on (t,x), we denote by $D_t h$ and $D_x h$ its partial derivatives. We use the symbol D without any subscript for the differentiation w.r.t. all variables except t, e.g. $Dh = D_{(u,y)}h$ for a function h that depends on (t,u,y). We use the same convention for higher order derivatives.

We identify the dual space of $\mathbb{R}^n$ with the space $\mathbb{R}^{n*}$ of n-dimensional horizontal vectors. Generally, we denote by X^* the dual space of a topological vector space X. Given a convex subset K of X and a point x of K, we denote by $T_K(x)$ and $N_K(x)$ the tangent and normal cone to K at x, respectively; see [25, Section 2.2.4] for their definition.

We denote by $|\cdot|$ both the Euclidean norm on finite-dimensional vector spaces and the cardinal of finite sets, and by $\|\cdot\|_s$ and $\|\cdot\|_{q,s}$ the standard norms on the Lesbesgue spaces L^s and the Sobolev spaces $W^{q,s}$, respectively.

We denote by $BV([0,T])$ the space of functions of bounded variation on the closed interval $[0,T]$. Any $h \in BV([0,T])$ has a derivative $\mathrm{d}h$ which is a finite Radon measure on $[0,T]$ and h_0 (resp. h_T) is defined by $h_0 := h_{0_+} - \mathrm{d}h(0)$ (resp. $h_T := h_{T_-} + \mathrm{d}h(T)$). Thus $BV([0,T])$ is endowed with the following norm: $\|h\|_{BV} := \|\mathrm{d}h\|_{\mathcal{M}} + |h_T|$. See [5, Section 3.2] for a rigorous presentation of BV.

All vector-valued inequalities have to be understood coordinate-wise.

1.2 Setting

1.2.1 The optimal control problem

Consider the *state equation*

$$\dot{y}_t = f(t,u_t,y_t) \quad \text{for a.a. } t \in (0,T). \tag{1.1}$$

Here, u is a *control* which belongs to $\mathcal{U}$, y is a *state* which belongs to $\mathcal{Y}$, where

$$\mathcal{U} := L^\infty(0,T;\mathbb{R}^m), \quad \mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n),$$

and $f\colon [0,T]\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}^n$ is the *dynamics*. Consider constraints of various types on the system: the *mixed control-state constraints*, or mixed constraints

$$c(t,u_t,y_t) \le 0 \quad \text{for a.a. } t \in (0,T), \tag{1.2}$$

the *pure state constraints*, or state constraints

$$g(t,y_t) \le 0 \quad \text{for a.a. } t \in (0,T), \tag{1.3}$$

and the *initial-final state constraints*

$$\begin{cases} \Phi^E(y_0, y_T) = 0, \\ \Phi^I(y_0, y_T) \leq 0. \end{cases} \tag{1.4}$$

Here $c\colon [0,T]\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}^{n_c}$, $g\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n_g}$, $\Phi^E : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n_{\Phi^E}}$, $\Phi^I : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n_{\Phi^I}}$. Consider finally the *cost function* $\phi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The *optimal control problem* is then

$$\min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \phi(y_0, y_T) \quad \text{subject to} \quad (1.1)\text{-}(1.4). \tag{P}$$

1.2.2 Definitions and assumptions

Similarly to [85, Definition 2.1], we introduce the following Carathéodory-type regularity notion:

Definition 1.2.1. We say that $\varphi\colon [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^s$ is *uniformly quasi-C^k* iff

(i) for a.a. t, $(u,y) \mapsto \varphi(t,u,y)$ is of class C^k, and the modulus of continuity of $(u,y) \mapsto D^k\varphi(t,u,y)$ on any compact of $\mathbb{R}^m \times \mathbb{R}^n$ is uniform w.r.t. t.

(ii) for $j = 0,\dots,k$, for all (u,y), $t \mapsto D^j\varphi(t,u,y)$ is essentially bounded.

Remark 1.2.2. If φ is uniformly quasi-C^k, then $D^j\varphi$ for $j = 0,\dots,k$ are essentially bounded on any compact, and $(u,y) \mapsto D^j\varphi(t,u,y)$ for $j = 0,\dots,k-1$ are locally Lipschitz, uniformly w.r.t. t. In particular, if f is uniformly quasi-C^1, then by Cauchy-Lipschitz theorem, for any $(u,y^0) \in \mathcal{U} \times \mathbb{R}^n$, there exists a unique $y \in \mathcal{Y}$ such that (1.1) holds and $y_0 = y^0$; we denote it by $y[u,y^0]$.

The minimal regularity assumption through all the paper is the following:

Assumption 1. The mappings f, c and g are uniformly quasi-C^1, g is continuous, Φ^E, Φ^I and ϕ are C^1.

We call a *trajectory* any pair $(u,y) \in \mathcal{U} \times \mathcal{Y}$ such that (1.1) holds. We say that a trajectory is *feasible* for problem (P) if it satisfies constraints (1.2)-(1.4), and denote by $F(P)$ the set of feasible trajectories. We define the *Hamiltonian* and the *augmented Hamiltonian* respectively by

$$H[p](t,u,y) := pf(t,u,y), \quad H^a[p,\nu](t,u,y) := pf(t,u,y) + \nu c(t,u,y),$$

for $(p,\nu,t,u,y) \in \mathbb{R}^{n*} \times \mathbb{R}^{n_c*} \times [0,T] \times \mathbb{R}^m \times \mathbb{R}^n$. We define the *end points Lagrangian* by

$$\Phi[\beta,\Psi](y_0,y_T) := \beta\phi(y_0,y_T) + \Psi\Phi(y_0,y_T),$$

for $(\beta,\Psi,y_0,y_T) \in \mathbb{R} \times \mathbb{R}^{n_\Phi*} \times \mathbb{R}^n \times \mathbb{R}^n$, where $n_\Phi = n_{\Phi^E} + n_{\Phi^I}$ and $\Phi = \begin{pmatrix} \Phi^E \\ \Phi^I \end{pmatrix}$.

We denote

$$K_c := L^\infty(0,T;\mathbb{R}^{n_c}_-), \quad K_g := C([0,T];\mathbb{R}^{n_g}_-), \quad K_\Phi := \{0\}_{\mathbb{R}^{n_{\Phi^E}}} \times \mathbb{R}^{n_{\Phi^I}}_-,$$

so that the constraints (1.2)-(1.4) can be rewritten as

$$c(\cdot,u,y) \in K_c, \quad g(\cdot,y) \in K_g, \quad \Phi(y_0,y_T) \in K_\Phi.$$

Recall that the dual space of $C([0,T];\mathbb{R}^{n_g})$ is the space $\mathcal{M}([0,T];\mathbb{R}^{n_g*})$ of finite vector-valued Radon measures. We denote by $\mathcal{M}([0,T];\mathbb{R}^{n_g*})_+$ the cone of positive measures in this dual space. Let

$$E := \mathbb{R} \times \mathbb{R}^{n_\Phi *} \times L^\infty(0,T;\mathbb{R}^{n_c*}) \times \mathcal{M}([0,T];\mathbb{R}^{n_g*}) \tag{1.5}$$

and let $\|\cdot\|_E$ be defined, for any $\lambda = (\beta, \Psi, \nu, \mu) \in E$, by

$$\|\lambda\|_E := |\beta| + |\Psi| + \|\nu\|_1 + \|\mu\|_{\mathcal{M}}. \tag{1.6}$$

Let $(\bar{u}, \bar{y}) \in F(P)$. Let N_{K_c} be the set of elements in the normal cone to K_c at $c(\cdot, \bar{u}, \bar{y})$ that belong to $L^\infty(0,T;\mathbb{R}^{n_c*})$, i.e.

$$N_{K_c}(c(\cdot,\bar{u},\bar{y})) := \left\{\nu \in L^\infty(0,T;\mathbb{R}^{n_c*}_+) \,:\, \nu_t c(t,\bar{u}_t,\bar{y}_t) = 0 \text{ for a.a. } t\right\}.$$

Let N_{K_g} be the normal cone to K_g at $g(\cdot,\bar{y})$, i.e.

$$N_{K_g}(g(\cdot,\bar{y})) := \left\{\mu \in \mathcal{M}([0,T];\mathbb{R}^{n_g*})_+ \,:\, \int_{[0,T]} (\mathrm{d}\mu_t g(t,\bar{y}_t)) = 0\right\}.$$

Let N_{K_Φ} be the normal cone to K_Φ at $\Phi(\bar{y}_0, \bar{y}_T)$, i.e.

$$N_{K_\Phi}(\Phi(\bar{y}_0,\bar{y}_T)) := \left\{\Psi \in \mathbb{R}^{n_\Phi *} \,:\, \begin{array}{l} \Psi_i \geq 0 \\ \Psi_i \Phi_i(\bar{y}_0,\bar{y}_T) = 0 \end{array} \text{ for } n_{\Phi^E} < i \leq n_\Phi \right\}.$$

Finally, let

$$N(\bar{u},\bar{y}) := \mathbb{R}_+ \times N_{K_\Phi}(\Phi(\bar{y}_0,\bar{y}_T)) \times N_{K_c}(c(\cdot,\bar{u},\bar{y})) \times N_{K_g}(g(\cdot,\bar{y})) \subset E.$$

We denote

$$\mathcal{P} := BV([0,T];\mathbb{R}^{n*}).$$

Given $(\bar{u},\bar{y}) \in F(P)$ and $\lambda = (\beta,\Psi,\nu,\mu) \in E$, we consider the *costate equation* in $\mathcal{P}$

$$\begin{cases} -\mathrm{d}p_t = D_y H^a[p_t,\nu_t](t,\bar{u}_t,\bar{y}_t)\mathrm{d}t + \mathrm{d}\mu_t Dg(t,\bar{y}_t), \\ \quad p_T = D_{y_T}\Phi[\beta,\Psi](\bar{y}_0,\bar{y}_T). \end{cases} \tag{1.7}$$

Lemma 1.2.3. *Let $(\bar{u},\bar{y}) \in F(P)$. For any $\lambda \in E$, there exists a unique solution of the costate equation* (1.7)*, that we denote by p^λ. The mapping*

$$\lambda \in E \mapsto p^\lambda \in \mathcal{P}$$

is linear continuous.

Proof. We first get the existence, uniqueness and the continuity of

$$\lambda \mapsto p^\lambda \in L^1(0,T;\mathbb{R}^{n*})$$

by a contraction argument. Then the continuity of

$$\lambda \mapsto (\mathrm{d}p, p_T) \in \mathcal{M}([0,T];\mathbb{R}^{n*}) \times \mathbb{R}^{n*}$$

follows by (1.7). □

Definition 1.2.4. Let $(\bar{u}, \bar{y}) \in F(P)$ and $\lambda = (\beta, \Psi, \nu, \mu) \in E$. We say that the solution of the costate equation (1.7) $p^\lambda \in \mathcal{P}$ is an *associated costate* iff

$$-p_0^\lambda = D_{y_0}\Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T).$$

Let $N_\pi(\bar{u}, \bar{y})$ be the set of *nonzero* $\lambda \in N(\bar{u}, \bar{y})$ having an associated costate.

Let $(\bar{u}, \bar{y}) \in F(P)$. We define the set-valued mapping $U\colon [0,T] \rightrightarrows \mathbb{R}^m$ by

$$U(t) := \operatorname{cl}\{u \in \mathbb{R}^m \,:\, c(t, u, \bar{y}_t) < 0\} \quad \text{for a.a. } t, \tag{1.8}$$

where cl denotes the closure in $\mathbb{R}^m$.

Definition 1.2.5. Let $(\bar{u}, \bar{y}) \in F(P)$. We say that the *inward condition* for the mixed constraints holds iff there exist $\gamma > 0$ and $\bar{v} \in \mathcal{U}$ such that

$$c(t, \bar{u}_t, \bar{y}_t) + D_u c(t, \bar{u}_t, \bar{y}_t)\bar{v}_t \le -\gamma, \quad \text{for a.a. } t.$$

Remark 1.2.6. If the inward condition holds, then there exists $\delta > 0$ such that, for a.a. t,

$$B_\delta(\bar{u}_t) \cap U(t) = B_\delta(\bar{u}_t) \cap \{u \in \mathbb{R}^m \,:\, c(t, u, \bar{y}_t) \le 0\},$$

where $B_\delta(\bar{u}_t)$ is the open ball in $\mathbb{R}^m$ of center $\bar{u}_t$ and radius δ. In particular, $\bar{u}_t \in U(t)$ for a.a. t.

In the sequel, we will always make the following assumption:

Assumption 2. The inward condition for the mixed constraints holds.

We can now define the notions of multipliers that we will consider. Recall that $N_\pi(\bar{u}, \bar{y})$ has been introduced in Definition 1.2.4.

Definition 1.2.7. Let $(\bar{u}, \bar{y}) \in F(P)$.

(i) We say that $\lambda \in N_\pi(\bar{u}, \bar{y})$ is a *generalized Lagrange multiplier* iff

$$D_u H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t) = 0 \quad \text{for a.a. } t.$$

We denote by $\Lambda_L(\bar{u}, \bar{y})$ the set of generalized Lagrange multipliers.

(ii) We say that $\lambda \in \Lambda_L(\bar{u}, \bar{y})$ is a *generalized Pontryagin multiplier* iff

$$H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) \le H[p_t^\lambda](t, u, \bar{y}_t) \quad \text{for all } u \in U(t), \quad \text{for a.a. } t. \tag{1.9}$$

We denote by $\Lambda_P(\bar{u}, \bar{y})$ the set of generalized Pontryagin multipliers.

(iii) We say that $\lambda \in \Lambda_P(\bar{u}, \bar{y})$ is a *degenerate Pontryagin equality multiplier* iff $\lambda = (\beta, \Psi, \nu, \mu)$ with $\Psi = (\Psi^E, \Psi^I)$ is such that $(\beta, \Psi^I, \nu, \mu) = 0$ and if equality holds in (1.9). We denote by $\Lambda_P^D(\bar{u}, \bar{y})$ the set of such multipliers.

Remark 1.2.8.

1. The sets $\Lambda_L(\bar{u}, \bar{y})$, $\Lambda_P(\bar{u}, \bar{y})$ and $\Lambda_P^D(\bar{u}, \bar{y})$ are positive cones of nonzero elements, possibly empty, and $\Lambda_P^D(\bar{u}, \bar{y})$ is symmetric.
2. Assumption 2 will be needed to get that the component ν of a multiplier, associated to the mixed constraints, belongs to $L^\infty(0, T; \mathbb{R}^{n_c *})$ and not only to $L^\infty(0, T; \mathbb{R}^{n_c})^*$. See [24, Theorem 3.1] and Theorem 1.A.4 in Appendix 1.A.1.

3. Let $\lambda \in \Lambda_P(\bar{u}, \bar{y})$. If Assumption 2 holds, then by Remark 1.2.6, $\bar{u}_t$ is a local solution of the finite dimensional optimization problem
$$\min_{u \in \mathbb{R}^m} H[p_t^\lambda](t, u, \bar{y}_t) \quad \text{subject to} \quad c(t, u, \bar{y}_t) \leq 0,$$
and ν_t is an associated Lagrange multiplier, for a.a. t.
4. See Appendix 1.A.4 for an example where there exists a multiplier such that (1.9) holds for all $u \in U(t)$, but not for all $u \in \{u \in \mathbb{R}^m : c(t, u, \bar{y}_t) \leq 0\}$.

We finish this section with various notions of minima, following [69].

Definition 1.2.9. We say that $(\bar{u}, \bar{y}) \in F(P)$ is a *global minimum* iff
$$\phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T) \quad \text{for all } (u, y) \in F(P), \tag{1.10}$$
a *Pontryagin minimum* iff for any $R > \|\bar{u}\|_\infty$, there exists $\varepsilon > 0$ such that
$$\phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T) \quad \begin{array}{l} \text{for all } (u, y) \in F(P) \text{ such that} \\ \|u - \bar{u}\|_1 + \|y - \bar{y}\|_\infty \leq \varepsilon \text{ and } \|u\|_\infty \leq R, \end{array} \tag{1.11}$$
a *weak minimum* iff there exists $\varepsilon > 0$ such that
$$\phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T) \quad \begin{array}{l} \text{for all } (u, y) \in F(P) \text{ such that} \\ \|u - \bar{u}\|_\infty + \|y - \bar{y}\|_\infty \leq \varepsilon. \end{array} \tag{1.12}$$

Remark 1.2.10. Obviously, (1.10) $\Rightarrow$ (1.11) $\Rightarrow$ (1.12). Conversely, if $(\bar{u}, \bar{y})$ is a weak minimum for problem (P), then it is a Pontryagin minimum for the problem obtained by adding the control constraint $|u_t - \bar{u}_t| \leq \varepsilon$, and a global minimum for the problem obtained by adding the same control constraint and the state constraint $|y_t - \bar{y}_t| \leq \varepsilon$.

1.3 First-order conditions in Pontryagin form

1.3.1 Pontryagin's minimum principle

First-order necessary conditions in Pontryagin form consist in proving the existence of Pontryagin multipliers. See Definitions 1.2.7 and 1.2.9 for the notions of multipliers and of minima. Our version of the well-known *Pontryagin's principle* follows, and is proved in Section 1.3.3. See [36] for a variant with the same approach, and [51] for a survey of this principle.

Theorem 1.3.1. *Let $(\bar{u}, \bar{y})$ be a Pontryagin minimum for problem (P) and let Assumptions 1-2 hold. Then the set of generalized Pontryagin multipliers $\Lambda_P(\bar{u}, \bar{y})$ is nonempty.*

By Remark 1.2.10, we get the following:

Corollary 1.3.2. *Let $(\bar{u}, \bar{y})$ be a weak minimum for problem (P) and let Assumptions 1-2 hold. Then there exist $\varepsilon > 0$ and $\lambda \in \Lambda_L(\bar{u}, \bar{y})$ such that*
$$\begin{cases} \text{for a.a. } t, \text{ for all } u \in U(t) \text{ such that } |u - \bar{u}_t| \leq \varepsilon, \\ H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) \leq H[p_t^\lambda](t, u, \bar{y}_t). \end{cases} \tag{1.13}$$

Proof. The extra control constraint $|u - \bar{u}_t| \leq \varepsilon$ for a.a. t is never active, therefore the set of Lagrange multipliers is unchanged. The set of Pontryagin multipliers is the set of Lagrange multipliers for which (1.13) holds. □

The proof of Theorem 1.3.1, given in Section 1.3.3, relies on first-order necessary conditions for a family of weak minima for auxiliary optimal control problems, namely the *partially relaxed problems*, presented in Section 1.3.2. These problems are defined using a *Castaing representation* of the set-valued mapping U, introduced at the beginning of Section 1.3.2. Second order necessary conditions in Pontryagin form in Section 1.4.1 will be derived from a variant of the partially relaxed problems, the reduced partially relaxed problems. Thus Section 1.3.2 is central. First and second order necessary conditions for a weak minimum are recalled, with some orginal results, in Appendix 1.A.1.

1.3.2 Partial relaxation

In this section, $(\bar{u}, \bar{y})$ is a given Pontryagin minimum for problem (P), and Assumptions 1-2 hold.

Castaing representation

See [29, 30, 81] for a general presentation of set-valued mappings and measurable selection theorems.

Definition 1.3.3. Let $V : [0, T] \rightrightarrows \mathbb{R}^m$ be a set-valued mapping. We say that a sequence $(v^k)_{k\in\mathbb{N}}$, $v^k \in \mathcal{U}$, is a *Castaing representation* of V iff $\{v_t^k\}_{k\in\mathbb{N}}$ is a dense subset of $V(t)$ for a.a. t.

Lemma 1.3.4. *There exists a Castaing representation $(u^k)_{k\in\mathbb{N}}$ of the set-valued mapping U defined by* (1.8)*, and for all k, there exists $\gamma_k > 0$ such that*

$$c(t, u_t^k, \bar{y}_t) \leq -\gamma_k \quad \textit{for a.a. } t.$$

Proof. For $l \in \mathbb{N}$, $l \geq 1$, we consider the set-valued mapping U_l defined by

$$U_l(t) := \left\{ u \in \mathbb{R}^n \,:\, c(t, u, \bar{y}_t) \leq -\tfrac{1}{l} \right\} \quad \text{for a.a. } t,$$

so that

$$U(t) = \operatorname{cl}\left(\cup_{l\geq 1} U_l(t)\right) \quad \text{for a.a. } t. \tag{1.14}$$

Under Assumptions 1-2, by [29, Théorème 3.5] and for l large enough, U_l is a measurable with nonempty closed set-valued mapping. Then by [29, Théorème 5.4], it has a Castaing representation. By (1.14), the union of such Castaing representations for l large enough is a Castaing representation of U. □

We define the following sequence of sets of generalized Lagrange multipliers: for $N \in \mathbb{N}$, let

$$\Lambda^N(\bar{u}, \bar{y}) := \left\{ \lambda \in \Lambda_L(\bar{u}, \bar{y}) \;:\; \begin{array}{l} H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) \leq H[p_t^\lambda](t, u_t^k, \bar{y}_t) \\ \text{for all } k \leq N, \text{ for a.a. } t \end{array} \right\}. \tag{1.15}$$

Observe that

$$\Lambda_P(\bar{u}, \bar{y}) \subset \Lambda^{N+1}(\bar{u}, \bar{y}) \subset \Lambda^N(\bar{u}, \bar{y}) \subset \Lambda_L(\bar{u}, \bar{y}),$$

and by density of the Castaing representation,

$$\Lambda_P(\bar{u},\bar{y}) = \bigcap_{N\in\mathbb{N}} \Lambda^N(\bar{u},\bar{y}). \tag{1.16}$$

Recall that E and $\|\cdot\|_E$ have been defined by (1.5) and (1.6).

Lemma 1.3.5. *Let $(\lambda^N)_{N\in\mathbb{N}}$ be a sequence in $\Lambda_L(\bar{u},\bar{y})$ such that $\|\lambda^N\|_E = 1$ and $\lambda^N \in \Lambda^N(\bar{u},\bar{y})$ for all N. Then the sequence has at least one nonzero weak $*$ limit point that belongs to $\Lambda_P(\bar{u},\bar{y})$.*

Proof. By Assumption 2 and [24, Theorem 3.1], the sequence is bounded in E for the usual norm, i.e. with $\|\nu\|_\infty$ instead of $\|\nu\|_1$. Then there exists $\bar{\lambda}$ such that, extracting a subsequence if necessary, $\lambda^N \rightharpoonup \bar{\lambda}$ for the weak $*$ topology. Since $N(\bar{u},\bar{y})$ is weakly $*$ closed, $\bar{\lambda} \in N(\bar{u},\bar{y})$. Observe now that if $\lambda \in N(\bar{u},\bar{y})$, then

$$\|\lambda\|_E = \beta + |\Psi| + \langle \nu, 1\rangle_1 + \langle \mu, 1\rangle_C$$

where $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_C$ are the dual products in $L^1(0,T;\mathbb{R}^{n_c})$ and $C([0,T];\mathbb{R}^{n_g})$, respectively, and the 1 are constant functions of appropriate size. Then $\|\bar{\lambda}\| = 1$ and $\bar{\lambda} \neq 0$. Let $p^N := p^{\lambda^N}$, $N \in \mathbb{N}$, and $\bar{p} := p^{\bar{\lambda}}$. By Lemma 1.2.3, $\mathrm{d}p^N \rightharpoonup \mathrm{d}\bar{p}$ for the weak $*$ topology in $\mathcal{M}([0,T];\mathbb{R}^{n*})$ and $p^N_T \to \bar{p}_T$. Since

$$p_0 = p_T - \langle \mathrm{d}p, 1\rangle_C$$

for any $p \in \mathcal{P}$, we derive that $\bar{p}_0 = D_{y_0}\Phi[\bar{\beta},\bar{\Psi}](\bar{y}_0,\bar{y}_T)$. Then $\bar{p}$ is an associated costate, i.e. $\bar{\lambda} \in N_\pi(\bar{u},\bar{y})$. Next, as a consequence of Lemma 1.2.3, $p^N \rightharpoonup \bar{p}$ for the weak $*$ topology in L^∞. Then $D_u H^a[p^N,\nu^N](\cdot,\bar{u},\bar{y}) \rightharpoonup D_u H^a[\bar{p},\bar{\nu}](\cdot,\bar{u},\bar{y})$ for the weak $*$ topology in L^∞, and then $D_u H^a[\bar{p}_t,\bar{\nu}_t](t,\bar{u}_t,\bar{y}_t) = 0$ for a.a. t, i.e. $\bar{\lambda} \in \Lambda_L(\bar{u},\bar{y})$. Similarly, for all $k \in \mathbb{N}$,

$$H[p^N](\cdot,u^k,\bar{y}) - H[p^N](\cdot,\bar{u},\bar{y}) \rightharpoonup H[\bar{p}](\cdot,u^k,\bar{y}) - H[\bar{p}](\cdot,\bar{u},\bar{y})$$

for the weak $*$ topology in L^∞, and then

$$H[\bar{p}_t](t,u^k_t,\bar{y}_t) - H[\bar{p}_t](t,\bar{u}_t,\bar{y}_t) \geq 0 \quad \text{for a.a. } t,$$

i.e. $\bar{\lambda} \in \Lambda^k(\bar{u},\bar{y})$, for all $k \in \mathbb{N}$. By (1.16), $\bar{\lambda} \in \Lambda_P(\bar{u},\bar{y})$. ☐

Since $\Lambda^N(\bar{u},\bar{y})$, $N \in \mathbb{N}$, are cones of nonzero elements (see Remark 1.2.8), it is enough to show that they are nonempty for all N to prove Theorem 1.3.1, by Lemma 1.3.5. This is the purpose of the *partially relaxed problems*, presented in the next section. Indeed, we will see that they are such that their Lagrange multipliers, whose existence can easily be guaranteed, belong to $\Lambda^N(\bar{u},\bar{y})$.

The partially relaxed problems

As motivated above, we introduce now a sequence of optimal control problems.

Formulation Recall that $(\bar{u},\bar{y})$ is given as a Pontryagin minimum for problem (P) has been given.

Let $N \in \mathbb{N}$. Consider the *partially relaxed state equation*

$$\dot{y}_t = \left(1 - \sum_{i=1}^N \alpha^i_t\right) f(t,u_t,y_t) + \sum_{i=1}^N \alpha^i_t f(t,u^i_t,y_t) \quad \text{for a.a. } t \in (0,T). \tag{1.17}$$

The u^i are elements of the Castaing representation given by Lemma 1.3.4. The controls are u and α, the state is y, with

$$u \in \mathcal{U}, \quad \alpha \in \mathcal{A}^N := L^\infty(0,T;\mathbb{R}^N), \quad y \in \mathcal{Y}.$$

The idea is to consider the problem of minimizing $\phi(y_0, y_T)$ under the same constraints as before, plus the control constraints $\alpha \geq 0$. To simplify the qualification issue, we actually introduce a *slack variable* $\theta \in \mathbb{R}$, with the intention to minimize it, and the following constraint on the cost function:

$$\phi(y_0, y_T) - \phi(\bar{y}_0, \bar{y}_T) \leq \theta. \tag{1.18}$$

The slack variable θ also enters into every inequality constraint:

$$\begin{aligned} -\alpha_t &\leq \theta \quad \text{for a.a. } t \in (0,T), && (1.19)\\ c(t, u_t, y_t) &\leq \theta \quad \text{for a.a. } t \in (0,T), && (1.20)\\ g(t, y_t) &\leq \theta \quad \text{for a.a. } t \in (0,T), && (1.21)\\ \Phi^I(y_0, y_T) &\leq \theta && (1.22) \end{aligned}$$

and the equality constraints remain unchanged:

$$\Phi^E(y_0, y_T) = 0. \tag{1.23}$$

The *partially relaxed problem* is

$$\min_{(u,\alpha,y,\theta)\in\mathcal{U}\times\mathcal{A}^N\times\mathcal{Y}\times\mathbb{R}} \theta \quad \text{subject to} \quad (1.17)\text{-}(1.23). \tag{P_N}$$

Let $\bar{\alpha} := 0 \in \mathcal{A}^N$ and $\bar{\theta} := 0 \in \mathbb{R}$. As for problem (P), we call a *relaxed trajectory* any (u, α, y, θ) such that (1.17) holds. We say that a relaxed trajectory is feasible if it satisties constraints (1.18)-(1.23), and denote by $F(P_N)$ the set of feasible relaxed trajectories.

Under Assumption 1, for any $(u, \alpha, y^0) \in \mathcal{U} \times \mathcal{A}^N \times \mathbb{R}^n$, there exists a unique $y \in \mathcal{Y}$ such that (1.17) holds and $y_0 = y^0$; we denote it by $y[u, \alpha, y^0]$ and consider the mapping

$$\Gamma_N \colon (u, \alpha, y^0) \mapsto y[u, \alpha, y^0]. \tag{1.24}$$

Remark 1.3.6. 1. We have $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta}) \in F(P_N)$.

2. Robinson's constraint qualification holds at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ iff the equality constraints are qualified, i.e. iff the derivative of

$$(u, \alpha, y^0) \in \mathcal{U} \times \mathcal{A}^N \times \mathbb{R}^n \mapsto \Phi^E(y^0, \Gamma_N(u, \alpha, y^0)_T) \in \mathbb{R}^{n_{\Phi^E}} \tag{1.25}$$

at $(\bar{u}, \bar{\alpha}, \bar{y}_0)$ is onto. We say that problem (P_N) is *qualified* at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ if this is the case. See [25, Section 2.3.4] for the definition and characterizations of Robinson's constraint qualification.

Existence of a minimum A key result is the following:

Theorem 1.3.7. *Let Assumptions 1-2 hold and let problem (P_N) be qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$. Then $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ is a weak minimum for this problem.*

Theorem 1.3.7 is a corollary of the following proposition, proved in the Appendix 1.A.2 for the sake of self-containment of the paper. It can also be deduced from other classical relaxation theorems, such as [37, Theorem 3].

Proposition 1.3.8. *Under the assumptions of Theorem 1.3.7, there exists $M > 0$ such that, for any $(\hat{u}, \hat{\alpha}, \hat{y}, \hat{\theta}) \in F(P_N)$ in a L^∞ neighborhood of $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ and with $\hat{\theta} < 0$, for any $\varepsilon > 0$, there exists $(\tilde{u}, \tilde{y}) \in F(P)$ such that*

$$\|\tilde{u} - \hat{u}\|_1 \leq M \|\hat{\alpha}\|_\infty \quad \text{and} \quad \|\tilde{y} - \hat{y}\|_\infty \leq \varepsilon.$$

Proof of Theorem 1.3.7. Suppose that $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ is not a weak minimum for problem (P_N). Then there exists $(\hat{u}, \hat{\alpha}, \hat{y}, \hat{\theta}) \in F(P_N)$ as L^∞ close to $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$ as needed and with $\hat{\theta} < 0$. Let $\varepsilon > 0$ be such that

$$\|y - \hat{y}\|_\infty \leq \varepsilon \;\Rightarrow\; \phi(y_0, y_T) < \phi(\bar{y}_0, \bar{y}_T).$$

By the proposition, we get $(\tilde{u}, \tilde{y}) \in F(P)$ such that $\phi(\tilde{y}_0, \tilde{y}_T) < \phi(\bar{y}_0, \bar{y}_T)$ and

$$\|\tilde{u} - \bar{u}\|_1 + \|\tilde{y} - \bar{y}\|_\infty \leq M \|\hat{\alpha}\|_\infty + T\|\hat{u} - \bar{u}\|_\infty + \varepsilon + \|\hat{y} - \bar{y}\|_\infty. \tag{1.26}$$

Observe that the right-hand side of (1.26) can be chosen as small as needed. Thus we get a contradiction with the Pontryagin optimality of $(\bar{u}, \bar{y})$. $\square$

Optimality conditions Problem (P_N) can be seen as an optimization problem over $(u, \alpha, y^0, \theta) \in \mathcal{U} \times \mathcal{A}^N \times \mathbb{R}^n \times \mathbb{R}$, via the mapping Γ_N defined by (1.24). Then we can define the set $\Lambda(P_N)$ of Lagrange multipliers at $(\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta})$ as in Appendix 1.A.1:

$$\Lambda(P_N) := \left\{(\lambda, \gamma) \in N(\bar{u}, \bar{y}) \times L^\infty(0, T; \mathbb{R}_+^{N*}) \,:\, DL_N[\lambda, \gamma](\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta}) = 0\right\}$$

where L_N is defined, for $\lambda = (\beta, \Psi, \nu, \mu)$, $\Psi = (\Psi^E, \Psi^I)$, $y = \Gamma_N(u, \alpha, y^0)$, by

$$\begin{aligned} L_N[\lambda, \gamma](u, \alpha, y^0, \theta) := \theta &+ \beta(\phi(y_0, y_T) - \phi(\bar{y}_0, \bar{y}_T) - \theta) \\ &+ \Psi^E \Phi^E(y_0, y_T) + \Psi^I(\Phi^I(y_0, y_T) - \theta) \\ &+ \int_{[0,T]} [\nu_t(c(t, u_t, y_t) - \theta)\mathrm{d}t + \mathrm{d}\mu_t(g(t, y_t) - \theta) - \gamma_t(\alpha_t + \theta)\mathrm{d}t]. \end{aligned} \tag{1.27}$$

In (1.27), θ has to be understood as a vector of appropriate size and with equal components. We have the following first-order necessary conditions:

Lemma 1.3.9. *Let problem (P_N) be qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$. Then $\Lambda(P_N)$ is nonempty, convex, and weakly $*$ compact.*

Proof. We apply Theorem 1.A.4 to $(\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta})$, locally optimal solution of (P_N) by Theorem 1.3.7. Let $\bar{v} \in \mathcal{U}$ be given by the inward condition for the mixed constraints in problem (P) (Assumption 2) and let $\bar{\omega} := 1 \in \mathcal{A}^N$. Then $(\bar{v}, \bar{\omega})$ satisfies the inward condition for the mixed constraints in problem (P_N). The other assumptions being also satisfied by Assumption 1 and Remark 1.3.6.2, the conclusion follows. $\square$

1.3.3 Proof of Theorem 1.3.1

As explained at the end of Section 1.3.2, it is enough by Lemma 1.3.5 to prove that $\Lambda^N(\bar{u}, \bar{y}) \neq \emptyset$ for all N. To do so, we use the partially relaxed problems (P_N) as follows:

Lemma 1.3.10. *Let* $(\lambda, \gamma) \in \Lambda(P_N)$. *Then* $\lambda \in \Lambda^N(\bar{u}, \bar{y})$.

Proof. Let (u, α, y, θ) be a relaxed trajectory and $(\lambda, \gamma) \in E \times L^\infty(0, T; \mathbb{R}^{N*})$, with $\lambda = (\beta, \Psi, \nu, \mu)$ and $\Psi = (\Psi^E, \Psi^I)$. Adding to L_N

$$0 = \int_0^T p_t \left(\left(1 - \sum \alpha_t^i\right) f(t, u_t, y_t) + \sum \alpha_t^i f(t, u_t^i, y_t) - \dot{y}_t \right) \mathrm{d}t,$$

and integrating by parts we have, for any $p \in \mathcal{P}$,

$$\begin{aligned} L_N[\lambda, \gamma](u, \alpha, y_0, \theta) &= \theta \left(1 - \beta - \left\langle \Psi^I, 1 \right\rangle - \langle \nu, 1 \rangle_1 - \langle \mu, 1 \rangle_C - \langle \gamma, 1 \rangle_1 \right) \\ &+ \int_0^T \left(H^a[p_t, \nu_t](t, u_t, y_t) + \sum_{i=1}^N \alpha_t^i \left(H[p_t](t, u_t^i, y_t) - H[p_t](t, u_t, y_t) - \gamma_t^i \right) \right) \mathrm{d}t \\ &+ \int_{[0,T]} (\mathrm{d}\mu_t g(t, y_t) + \mathrm{d}p_t y_t) + \Phi[\beta, \Psi](y_0, y_T) - p_T y_T + p_0 y_0 - \beta \phi(\bar{y}_0, \bar{y}_T). \end{aligned} \tag{1.28}$$

Let $(\lambda, \gamma) \in \Lambda(P_N)$. Using the expression (1.28) of L_N, we get

$$D_{y_0} \Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T) + p_0^\lambda = 0, \tag{1.29}$$
$$D_u H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t) = 0 \quad \text{for a.a. } t, \tag{1.30}$$
$$H[p_t^\lambda](t, u_t^i, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) = \gamma_t^i \quad \text{for a.a. } t, \quad 1 \leq i \leq N, \tag{1.31}$$
$$\beta + \left\langle \Psi^I, 1 \right\rangle + \langle \nu, 1 \rangle_1 + \langle \mu, 1 \rangle_C + \langle \gamma, 1 \rangle_1 = 1. \tag{1.32}$$

Suppose that $\lambda = 0$. Then $p^\lambda = 0$ and by (1.31), $\gamma = 0$; we get a contradiction with (1.32). Then $\lambda \neq 0$ and $\lambda \in N_\pi(\bar{u}, \bar{y})$ by (1.29). Finally, $\lambda \in \Lambda_L(\bar{u}, \bar{y})$ by (1.30), and $\lambda \in \Lambda^N(\bar{u}, \bar{y})$ by (1.31) since $\gamma \in L^\infty(0, T; \mathbb{R}_+^{N*})$. ☐

We need one more lemma:

Lemma 1.3.11. *Let problem* (P_N) *be not qualified at* $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$. *Then there exists* $\lambda \in \Lambda^N(\bar{u}, \bar{y})$ *such that* $-\lambda \in \Lambda^N(\bar{u}, \bar{y})$ *too, and for all* $k \leq N$,

$$H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) = H[p_t^\lambda](t, u_t^i, \bar{y}_t) \quad \textit{for a.a. } t. \tag{1.33}$$

Proof. Recall that Γ_N has been defined by (1.24). By Remark 1.3.6.2, there exists $\Psi^E \neq 0$ such that

$$\Psi^E D\Phi^E(\bar{y}_0, \bar{y}_T) D\Gamma_N(\bar{u}, \bar{\alpha}, \bar{y}_0) = 0.$$

Let $\Psi = (\Psi^E, 0)$ and $\lambda := \big(0, \Psi, 0, 0\big)$, so that $D_{(u,\alpha,y^0)} L_N[\lambda, 0](\bar{u}, \bar{\alpha}, \bar{y}_0, \bar{\theta}) = 0$ by (1.27). By (1.28), we get

$$D_{y_0} \Phi[0, (\Psi^E, 0)](\bar{y}_0, \bar{y}_T) + p_0^\lambda = 0,$$
$$D_u H^a[p_t^\lambda, 0](t, \bar{u}_t, \bar{y}_t) = 0 \quad \text{for a.a. } t,$$
$$H[p_t^\lambda](t, u_t^i, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) = 0 \quad \text{for a.a. } t, \quad 1 \leq i \leq N.$$

Then $\lambda \in \Lambda^N(\bar{u}, \bar{y})$ and (1.33) holds. ☐

We can now conclude:

Proof of Theorem 1.3.1. We need $\Lambda^N(\bar{u},\bar{y}) \neq \emptyset$ for all N. If problem (P_N) is qualified at $(\bar{u},\bar{\alpha},\bar{y},\bar{\theta})$, then $\Lambda^N(\bar{u},\bar{y}) \neq \emptyset$ by Lemmas 1.3.9 and 1.3.10. If problem (P_N) is not qualified at $(\bar{u},\bar{\alpha},\bar{y},\bar{\theta})$, then $\Lambda^N(\bar{u},\bar{y}) \neq \emptyset$ by Lemma 1.3.11. □

Actually, we have the following alternative:

Corollary 1.3.12. *The partially relaxed problems (P_N) are either qualified for all N large enough, if $\Lambda_P^D(\bar{u},\bar{y}) = \emptyset$, or never qualified, and then $\Lambda_P^D(\bar{u},\bar{y}) \neq \emptyset$.*

Proof. If the problems (P_N) are never qualified, then we get a sequence a multipliers as in the proof of Lemma 1.3.11. By the proof of Lemma 1.3.5, its limit points belong to $\Lambda_P^D(\bar{u},\bar{y})$. □

See Appendix 1.A.3 for a qualification condition ensuring the non singularity of the generalized Pontryagin multipliers.

1.4 Second-order conditions in Pontryagin form

1.4.1 Statement

The second-order necessary conditions presented in this section involve Pontryagin multipliers only. They rely again on the partially relaxed problems, introduced in Section 1.3.2. These problems are actually modified into *reduced* partially relaxed problems, which satisfy an *extended polyhedricity condition*, [25, Section 3.2.3]. The idea is to get our second-order necessary conditions on a large cone by density of the so-called *strict radial critical cone*, so that we do not have to compute the *envelope-like effect* of Kawasaki [58].

The main result of this section is Theorem 1.4.9. It is stated after some new definitions and assumptions, and proved in Section 1.4.2.

Definitions and assumptions

For second-order optimality conditions, we need a stronger regularity assumption than Assumption 1. Namely, we make in the sequel the following:

Assumption 3. The mappings f and g are C^∞, c is uniformly quasi-C^2, Φ and ϕ are C^2.

Remark 1.4.1. If there is no pure state constraint in problem (P) (i.e. no mapping g), we will see that it is enough to assume that f is uniformly quasi-C^2.

For $s \in [1,\infty]$, let

$$\mathcal{V}_s := L^s(0,T;\mathbb{R}^m), \quad \mathcal{Z}_s := W^{1,s}(0,T;\mathbb{R}^n).$$

Let $(\bar{u},\bar{y})$ be a trajectory for problem (P). Given $v \in \mathcal{V}_s$, $s \in [1,\infty]$, we consider the *linearized state equation* in $\mathcal{Z}_s$

$$\dot{z}_t = Df(t,\bar{u}_t,\bar{y}_t)(v_t,z_t) \quad \text{for a.a. } t \in (0,T). \tag{1.34}$$

We call a *linearized trajectory* any $(v,z) \in \mathcal{V}_s \times \mathcal{Z}_s$ such that (1.34) holds. For any $(v,z^0) \in \mathcal{V}_s \times \mathbb{R}^n$, there exists a unique $z \in \mathcal{Z}_s$ such that (1.34) holds and $z_0 = z^0$; we denote it by $z = z[v,z^0]$.

For $1 \leq i \leq n_g$, we define $g_i^{(j)} \colon [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, $j \in \mathbb{N}$, recursively by

$$g_i^{(j+1)}(t,u,y) := D_t g_i^{(j)}(t,u,y) + D_y g_i^{(j)}(t,u,y) f(t,u,y), \quad g_i^{(0)} := g_i.$$

Definition 1.4.2. The *order of a state constraint* g_i is $q_i \in \mathbb{N}$ such that

$$D_u g_i^{(j)} \equiv 0 \quad \text{for } 0 \leq j \leq q_i - 1, \quad D_u g_i^{(q_i)} \not\equiv 0.$$

Remark 1.4.3. If g_i is of order q_i, then $t \mapsto g_i(t, \bar{y}_t) \in W^{q_i,\infty}(0,T)$ for any trajectory $(\bar{u}, \bar{y})$, and

$$\begin{aligned}
\frac{\mathrm{d}^j}{\mathrm{d}t^j} g_i(t, \bar{y}_t) &= g_i^{(j)}(t, \bar{y}_t) \quad \text{for } 0 \leq j \leq q_i - 1, \\
\frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}} g_i(t, \bar{y}_t) &= g_i^{(q_i)}(t, \bar{u}_t, \bar{y}_t).
\end{aligned}$$

We have the same regularity along linearized trajectories; the proof of the next lemma is classical, see for instance [20, Lemma 9].

Lemma 1.4.4. *Let $(\bar{u}, \bar{y})$ be a trajectory and $(v, z) \in \mathcal{V}_s \times \mathcal{Z}_s$ be a linearized trajectory, $s \in [1, \infty]$. Let the constraint g_i be of order q_i. Then*

$$t \mapsto Dg_i(t, \bar{y}_t) z_t \in W^{q_i,s}(0,T),$$

and

$$\begin{aligned}
\frac{\mathrm{d}^j}{\mathrm{d}t^j} Dg_i(t, \bar{y}_t) z_t &= Dg_i^{(j)}(t, \bar{y}_t) z_t \quad \textit{for } 0 \leq j \leq q_i - 1, \\
\frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}} Dg_i(t, \bar{y}_t) z_t &= Dg_i^{(q_i)}(t, \bar{u}_t, \bar{y}_t)(v_t, z_t).
\end{aligned}$$

Definition 1.4.5. Let $(\bar{u}, \bar{y}) \in F(P)$. We say that $\tau \in [0,T]$ is a *touch point* for the constraint g_i iff it is a contact point for g_i, i.e. $g_i(\tau, \bar{y}_\tau) = 0$, isolated in $\{t : g_i(t, \bar{y}_t) = 0\}$. We say that a touch point τ for g_i is *reducible* iff $\tau \in (0,T)$, $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(t, \bar{y}_t)$ is defined for t close to τ, continuous at τ, and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(t, \bar{y}_t)|_{t=\tau} < 0.$$

Remark 1.4.6. If g_i is of order at least 2, then by Remark 1.4.3 a touch point τ for g_i is reducible iff $t \mapsto g_i^{(2)}(t, \bar{u}_t, \bar{y}_t)$ is continuous at τ and $g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau) < 0$. The continuity holds if $\bar{u}$ is continuous at τ or if g_i is of order at least 3.

Let $(\bar{u}, \bar{y}) \in F(P)$. For $1 \leq i \leq n_g$, let

$$\begin{aligned}
\mathcal{T}_{g,i} &:= \begin{cases} \emptyset & \text{if } g_i \text{ is of order 1}, \\ \{\text{touch points for } g_i\} & \text{if } g_i \text{ is of order at least 2}, \end{cases} \\
\Delta_{g,i}^0 &:= \{t \in [0,T] : g_i(t, \bar{y}_t) = 0\} \setminus \mathcal{T}_{g,i}, \\
\Delta_{g,i}^\varepsilon &:= \{t \in [0,T] : \operatorname{dist}(t, \Delta_{g,i}^0) \leq \varepsilon\},
\end{aligned}$$

and for $1 \leq i \leq n_c$, let

$$\Delta_{c,i}^\delta := \{t \in [0,T] : c_i(t, \bar{u}_t, \bar{y}_t) \geq -\delta\}.$$

We will need the following two extra assumptions:

Assumption 4. For $1 \leq i \leq n_g$, the set $\mathcal{T}_{g,i}$ is finite and contains only reducible touch points, $\Delta_{g,i}^0$ has finitely many connected components and g_i is of finite order q_i.

Assumption 5. There exist $\delta', \varepsilon' > 0$ such that the linear mapping from $\mathcal{V}_2 \times \mathbb{R}^n$ to $\prod_{i=1}^{n_c} L^2(\Delta_{c,i}^{\delta'}) \times \prod_{i=1}^{n_g} W^{q_i,2}(\Delta_{g,i}^{\varepsilon'})$ defined by

$$(v, z^0) \mapsto \begin{pmatrix} \left(Dc_i(\cdot, \bar{u}, \bar{y})(v, z[v, z^0])|_{\Delta_{c,i}^{\delta'}} \right)_{1 \leq i \leq n_c} \\ \left(Dg_i(\cdot, \bar{y}) z[v, z^0]|_{\Delta_{g,i}^{\varepsilon'}} \right)_{1 \leq i \leq n_g} \end{pmatrix} \quad \text{is onto.}$$

Remark 1.4.7. There exist sufficient conditions, of linear independance type, for Assumption 5 to hold. See for instance [22, Lemma 2.3] or [17, Lemma 4.5].

Main result

Let $(\bar{u}, \bar{y}) \in F(P)$. We define the *critical cone* in L^2

$$C_2(\bar{u}, \bar{y}) := \left\{ \begin{array}{l} (v, z) \in \mathcal{V}_2 \times \mathcal{Z}_2 \ : \ z = z[v, z_0] \\ D\phi(\bar{y}_0, \bar{y}_T)(z_0, z_T) \leq 0 \\ D\Phi(\bar{y}_0, \bar{y}_T)(z_0, z_T) \in T_{K_\Phi}(\Phi(\bar{y}_0, \bar{y}_T)) \\ Dc(\cdot, \bar{u}, \bar{y})(v, z) \in T_{K_c}(c(\cdot, \bar{u}, \bar{y})) \\ Dg(\cdot, \bar{y}) z \in T_{K_g}(g(\cdot, \bar{y})) \end{array} \right\}$$

and the *strict critical cone* in L^2

$$C_2^S(\bar{u}, \bar{y}) := \left\{ \begin{array}{lll} (v, z) \in C_2(\bar{u}, \bar{y}) \ : & & \\ Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) = 0 & t \in \Delta_{c,i}^0 & 1 \leq i \leq n_c \\ Dg_i(t, \bar{y}_t) z_t = 0 & t \in \Delta_{g,i}^0 & 1 \leq i \leq n_g \end{array} \right\}. \tag{1.35}$$

Remark 1.4.8.

1. See [25, Examples 2.63 and 2.64] for the description of T_{K_g} and T_{K_c}, respectively.
2. Since by Assumption 4 there are finitely many touch points for constraints of order at least 2, $C_2^S(\bar{u}, \bar{y})$ is defined by equality constraints and a finite number of inequality constraints, i.e. the cone $C_2^S(\bar{u}, \bar{y})$ is a polyhedron.
3. The strict critical cone $C_2^S(\bar{u}, \bar{y})$ is a subset of the critical cone $C_2(\bar{u}, \bar{y})$. But if there exists $\lambda = (\bar{\beta}, \bar{\Psi}, \bar{\nu}, \bar{\mu}) \in \Lambda_L(\bar{u}, \bar{y})$ such that

$$\begin{array}{ll} \bar{\nu}_i(t) > 0 \quad \text{for a.a. } t \in \Delta_{c,i}^0 & 1 \leq i \leq n_c, \\ \Delta_{g,i}^0 \subset \operatorname{supp}(\bar{\mu}_i) & 1 \leq i \leq n_g, \end{array}$$

then $C_2^S(\bar{u}, \bar{y}) = C_2(\bar{u}, \bar{y})$ (see [25, Proposition 3.10]).

For any $\lambda = (\beta, \Psi, \nu, \mu) \in E$, we define a quadratic form, the *Hessian of Lagrangian*, $\Omega[\lambda] \colon \mathcal{V}_2 \times \mathcal{Z}_2 \to \mathbb{R}$ by

$$\begin{aligned} \Omega[\lambda](v, z) := & \int_0^T D^2 H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)(v_t, z_t)^2 \mathrm{d}t + D^2\Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T)^2 \\ & + \int_{[0,T]} \mathrm{d}\mu_t D^2 g(t, \bar{y}_t)(z_t)^2 - \sum_{\substack{\tau \in \mathcal{T}_{g,i} \\ 1 \leq i \leq n_g}} \mu_i(\{\tau\}) \frac{\left(Dg_i^{(1)}(\tau, \bar{y}_\tau) z_\tau \right)^2}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau)}. \end{aligned}$$

We can now state our main result, that will be proved in the next section.

Theorem 1.4.9. *Let $(\bar{u}, \bar{y})$ be a Pontryagin minimum for problem (P) and let Assumptions 2-5 hold. Then for any $(v, z) \in C_2^S(\bar{u}, \bar{y})$, there exists $\lambda \in \Lambda_P(\bar{u}, \bar{y})$ such that*

$$\Omega[\lambda](v, z) \geq 0.$$

Remark 1.4.10. If $\Lambda_P^D(\bar{u}, \bar{y}) \neq \emptyset$ and $\lambda \in \Lambda_P^D(\bar{u}, \bar{y})$, then $-\lambda \in \Lambda_P^D(\bar{u}, \bar{y})$ too. Since $\Omega[-\lambda](v, z) = -\Omega[\lambda](v, z)$ for any $(v, z) \in \mathcal{V}_2 \times \mathcal{Z}_2$, Theorem 1.4.9 is then pointless. See Corollary 1.3.12 about the emptiness of $\Lambda_P^D(\bar{u}, \bar{y})$.

1.4.2 Proof of Theorem 1.4.9

In this section, $(\bar{u}, \bar{y})$ is a given Pontryagin minimum for problem (P), and Assumptions 2-5 hold.

Reduction and partial relaxation

The reduction approach [20, section 5] consists in reformulating the state constraint in the neighborhood of a touch point, using its reducibility (Definition 1.4.5). We apply this approach to the partially relaxed problems (P_N) in order to involve Pontryagin multipliers (see Lemmas 1.3.5 and 1.3.10).

Let $N \in \mathbb{N}$. Recall that Γ_N has been defined by (1.24).

Remark 1.4.11. The result of Remark 1.4.3 still holds for relaxed trajectories:

$$t \mapsto g_i(t, y_t) \in W^{q_i,\infty}(0, T) \quad \text{for any } y = \Gamma_N(u, \alpha, y^0).$$

Let $\tau \in \mathcal{T}_{g,i}$. We define $\Theta_{i,\tau}^{\varepsilon,N} : \mathcal{U} \times \mathcal{A}^N \times \mathbb{R}^n \to \mathbb{R}$ by

$$\Theta_{i,\tau}^{\varepsilon,N}(u, \alpha, y^0) := \max \left\{ g_i(t, y_t) \, : \, y = \Gamma_N(u, \alpha, y^0), \, t \in [\tau - \varepsilon, \tau + \varepsilon] \cap [0, T] \right\}.$$

Let $\bar{\Gamma}'_N := D\Gamma_N(\bar{u}, \bar{\alpha}, \bar{y}_0)$ and $\bar{\Gamma}''_N := D^2\Gamma_N(\bar{u}, \bar{\alpha}, \bar{y}_0)$.

Remark 1.4.12. Let $\bar{\omega} := 0 \in \mathcal{A}^N$. For any $(v, z^0) \in \mathcal{V}_s \times \mathbb{R}^n$, $s \in [1, \infty]$, we have

$$\bar{\Gamma}'_N(v, \bar{\omega}, z^0) = z[v, z^0].$$

Lemma 1.4.13. *There exists $\varepsilon > 0$ independent of N such that for any $\tau \in \mathcal{T}_{g,i}$, $\Theta_{i,\tau}^{\varepsilon,N}$ is C^1 in a neighborhood of $(\bar{u}, \bar{\alpha}, \bar{y}_0)$ and twice Fréchet differentiable at $(\bar{u}, \bar{\alpha}, \bar{y}_0)$, with first and second derivatives given by*

$$D\Theta_{i,\tau}^{\varepsilon,N}(\bar{u}, \bar{\alpha}, \bar{y}_0)(v, \omega, z^0) = Dg_i(\tau, \bar{y}_\tau)\bar{\Gamma}'_N(v, \omega, z^0)_\tau$$

for any $(v, \omega, z^0) \in \mathcal{V}_1 \times L^1(0, T; R^N) \times \mathbb{R}^n$, and

$$\begin{aligned} D^2\Theta_{i,\tau}^{\varepsilon,N}(\bar{u}, \bar{\alpha}, \bar{y}_0)(v, \omega, z^0)^2 &= D^2 g_i(\tau, \bar{y}_\tau)\left(\bar{\Gamma}'_N(v, \omega, z^0)_\tau\right)^2 \\ &\quad + Dg_i(\tau, \bar{y}_\tau)\bar{\Gamma}''_N(v, \omega, z^0)^2_\tau - \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t} Dg_i(\cdot, \bar{y})\bar{\Gamma}'_N(v, \omega, z^0)|_\tau\right)^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(\cdot, \bar{y})|_\tau} \end{aligned}$$

for any $(v, \omega, z^0) \in \mathcal{V}_2 \times L^2(0, T; R^N) \times \mathbb{R}^n$.

Proof. Combine [20, Lemma 23] with Remark 1.4.11 and Assumption 4. □

The reduced partially relaxed problems The formulation is the same as for problems (P_N), except that (i) we localize the mixed constraints c and the state constraints g on the domains given by Assumption 5, (ii) we replace the state constraints of order at least 2 around their touch points with the mappings $\Theta_{i,\tau}^{\varepsilon,N}$. Without loss of generality we assume that ε' given by Assumption 5 is smaller than ε given by Lemma 1.4.13; δ' is also given by Assumption 5.

Let $N \in \mathbb{N}$. Recall that in Section 1.3.2 the partially relaxed problem was

$$\min_{(u,\alpha,y,\theta)\in\mathcal{U}\times\mathcal{A}^N\times\mathcal{Y}\times\mathbb{R}} \theta \quad \text{subject to} \quad (1.17)\text{-}(1.23). \tag{P_N}$$

We consider the following new constraints:

$$c_i(t,u_t,y_t) \le \theta \quad \text{for a.a. } t \in \Delta_{c,i}^{\delta'} \quad 1 \le i \le n_c, \tag{1.36}$$

$$g_i(t,y_t) \le \theta \quad \text{for a.a. } t \in \Delta_{g,i}^{\varepsilon'} \quad 1 \le i \le n_g, \tag{1.37}$$

$$\Theta_{i,\tau}^{\varepsilon',N}(u,\alpha,y_0) \le \theta \quad \text{for all } \tau \in \mathcal{T}_{g,i} \quad 1 \le i \le n_g. \tag{1.38}$$

The *reduced partially relaxed problem* is then

$$\min_{(u,\alpha,y,\theta)\in\mathcal{U}\times\mathcal{A}^N\times\mathcal{Y}\times\mathbb{R}} \theta \quad \text{subject to} \quad (1.17)\text{-}(1.19), (1.22)\text{-}(1.23), (1.36)\text{-}(1.38). \tag{P_N^R}$$

As before, we denote by $F(P_N^R)$ the set of feasible relaxed trajectories.

Remark 1.4.14.
1. We have $(\bar{u},\bar{\alpha},\bar{y},\bar{\theta}) \in F(P_N^R)$ and, in a neighborhood of $(\bar{u},\bar{\alpha},\bar{y},\bar{\theta})$, $(u,\alpha,y,\theta) \in F(P_N^R)$ iff $(u,\alpha,y,\theta) \in F(P_N)$. In particular, $(\bar{u},\bar{\alpha},\bar{y},\bar{\theta})$ is a weak minimum for problem (P_N^R) iff it is a weak minimum for problem (P_N).
2. Problem (P_N^R) is qualified at $(\bar{u},\bar{\alpha},\bar{y},\bar{\theta})$ iff problem (P_N) is qualified at $(\bar{u},\bar{\alpha},\bar{y},\bar{\theta})$ (see Remark 1.3.6.2).

Optimality conditions Again, problem (P_N^R) can be seen as an optimization problem over (u,α,y^0,θ), via the mapping Γ_N. We denote its Lagrangian by L_N^R, its set of Lagrange multipliers at $(\bar{u},\bar{\alpha},\bar{y}_0,\bar{\theta})$ by $\Lambda(P_N^R)$, and its set of quasi radial critical directions in L^2 by $C_2^{QR}(P_N^R)$, as defined in Appendix 1.A.1.

Remark 1.4.15. By Lemma 1.4.13, we can identify $\Lambda(P_N^R)$ and $\Lambda(P_N)$ by identifying the scalar components of a multiplier associated to the constraints (1.38) and Dirac measures. See also [20, Lemma 26] or [17, Lemma 3.4].

We have the following second-order necessary conditions:

Lemma 1.4.16. *Let problem (P_N^R) be qualified at $(\bar{u},\bar{\alpha},\bar{y},\bar{\theta})$. Then for any $(v,\omega,z^0,\vartheta) \in \mathrm{cl}\left(C_2^{QR}(P_N^R)\right)$, there exists $(\lambda,\gamma) \in \Lambda(P_N^R)$ such that*

$$D^2 L_N^R[\lambda,\gamma](v,\omega,z^0,\vartheta)^2 \ge 0.$$

Here, cl *denotes the L^2 closure.*

Proof. We apply Theorem 1.A.5 to $(\bar{u},\bar{\alpha},\bar{y}_0,\bar{\theta})$, locally optimal solution of (P_N^R) by Theorem 1.3.7 and Remark 1.4.14. The various mappings have the required regularity by Assumption 3 and Lemma 1.4.13. Robinson's contraint qualification and the inward condition for the mixed constraints hold as in the proof of Lemma 1.3.9. The conclusion follows. ☐

Proof of the theorem

Let $(\bar{v}, \bar{z}) \in C_2^S(\bar{u}, \bar{y})$. By Lemma 1.3.5 and since $\lambda \mapsto \Omega[\lambda](\bar{v}, \bar{z})$ is linear continuous, it is enough to show that for all N, there exists $\lambda^N \in \Lambda^N(\bar{u}, \bar{y})$ such that

$$\Omega[\lambda^N](\bar{v}, \bar{z}) \geq 0. \tag{1.39}$$

Let $(\bar{\omega}, \bar{\vartheta}) := (0, 0) \in \mathcal{A}^N \times \mathbb{R}$. The link with the reduced partially relaxed problems (P_N^R) is as follows:

Lemma 1.4.17. *Let $(\lambda, \gamma) \in \Lambda(P_N^R)$. Then $\lambda \in \Lambda^N(\bar{u}, \bar{y})$ and*

$$D^2 L_N^R[\lambda, \gamma](\bar{v}, \bar{\omega}, \bar{z}_0, \bar{\vartheta})^2 = \Omega[\lambda](\bar{v}, \bar{z}).$$

Proof. The first part of the result is known by Lemma 1.3.10 and Remark 1.4.15. For the second part, we write L_N^R using H^a and H, as in the expression (1.28) of L_N, and we compute its second derivative. The result follows by Lemma 1.4.13 and Remark 1.4.12. See also [20, Lemma 26] or [17, Lemma 3.5]. □

We also need the following density result, that will be proved in Section 1.4.2.

Lemma 1.4.18. *The direction $(\bar{v}, \bar{\omega}, \bar{z}_0, \bar{\vartheta})$ belongs to $\mathrm{cl}\left(C_2^{QR}(P_N^R)\right)$, the closure of the set of quasi radial critical directions in L^2.*

We can now conclude:

Proof of Theorem 1.4.9. We need $\lambda^N \in \Lambda^N(\bar{u}, \bar{y})$ such that (1.39) holds for all N. If problem (P_N^R) is qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$, then we get λ^N as needed by Lemmas 1.4.16, 1.4.17 and 1.4.18. If problem (P_N^R) is not qualified at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$, then we get λ such that $-\lambda, \lambda \in \Lambda^N(\bar{u}, \bar{y})$ by Remark 1.4.14.2 and Lemma 1.3.11. Since $\lambda \mapsto \Omega[\lambda](\bar{v}, \bar{z})$ is linear, (1.39) holds for $\lambda^N = \pm\lambda$. □

A density result

In this section we prove Lemma 1.4.18. Recall that δ' is given by Assumption 5. We define the *strict radial critical cone* in L^2

$$C_2^R(\bar{u}, \bar{y}) := \left\{ \begin{array}{lll} (v, z) \in C_2(\bar{u}, \bar{y}) \ : & & \\ \exists \delta > 0 \ : \ Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t) = 0 & t \in \Delta_{c,i}^{\delta} & 1 \leq i \leq n_c \\ \exists M > 0 \ : \ |Dc_i(t, \bar{u}_t, \bar{y}_t)(v_t, z_t)| \leq M & t \in \Delta_{c,i}^{\delta'} & 1 \leq i \leq n_c \\ \exists \varepsilon > 0 \ : \ Dg_i(t, \bar{y}_t) z_t = 0 & t \in \Delta_{g,i}^{\varepsilon} & 1 \leq i \leq n_g \end{array} \right\}. \tag{1.40}$$

Proposition 1.4.19. *The strict radial critical cone $C_2^R(\bar{u}, \bar{y})$ is a dense subset of the strict critical cone $C_2^S(\bar{u}, \bar{y})$.*

Proof. Touch points for g_i are included in $\Delta_{g,i}^{\varepsilon}$, $\varepsilon \geq 0$, iff g_i is of order 1.

(a) Let $W^{(q),2}(0,T) := \prod_{i=1}^{n_g} W^{q_i,2}(0,T)$. We claim that the subspace

$$\left\{ \begin{array}{lll} (\phi, \psi) \in L^\infty(0, T; \mathbb{R}^{n_c}) \times W^{(q),2}(0,T) \ : & & \\ \exists \delta > 0 \ : \ \phi_{i,t} = 0 & t \in \Delta_{c,i}^{\delta} & 1 \leq i \leq n_c \\ \exists \varepsilon > 0 \ : \ \psi_{i,t} = 0 & t \in \Delta_{g,i}^{\varepsilon} & 1 \leq i \leq n_g \end{array} \right\}$$

is a dense subset of

$$\left\{ \begin{array}{c} (\phi,\psi) \in L^2(0,T;\mathbb{R}^{n_c}) \times W^{(q),2}(0,T) \ : \\ \phi_{i,t} = 0 \quad t \in \Delta^0_{c,i} \quad 1 \le i \le n_c \\ \psi_{i,t} = 0 \quad t \in \Delta^0_{g,i} \quad 1 \le i \le n_g \end{array} \right\}.$$

Indeed, for $\phi_i \in L^2(0,T)$, we consider the sequence

$$\phi^k_{i,t} := \begin{cases} 0 & \text{if } t \in \Delta^{1/k}_{c,i}, \\ \min\{k, |\phi_{i,t}|\} \frac{\phi_{i,t}}{|\phi_{i,t}|} & \text{otherwise.} \end{cases}$$

For $\psi_i \in W^{q_i,2}(0,T)$, we use the fact that there is no isolated point in $\Delta^0_{g,i}$ if $q_i \ge 2$, and approximation results in $W^{q_i,2}(0,T)$, e.g. [17, Appendix A.3]. Our claim follows.

(b) By Assumption 5 and the open mapping theorem, there exists $C > 0$ such that for all $(\phi,\psi) \in L^2(0,T;\mathbb{R}^{n_c}) \times W^{(q),2}(0,T)$, there exists $(v,z) \in \mathcal{V}_2 \times \mathcal{Z}_2$ such that

$$\begin{aligned} & z = z[v,z_0], \quad \|v\|_2 + |z_0| \le C \left(\|\phi\|_2 + \|\psi\|_{(q),2} \right), \\ & Dc_i(t,\bar{u}_t,\bar{y}_t)(v_t,z_t) = \phi_{i,t} \quad t \in \Delta^{\delta'}_{c,i} \quad 1 \le i \le n_c, \\ & Dg_i(t,\bar{y}_t)z_t = \psi_{i,t} \qquad\qquad\quad t \in \Delta^{\varepsilon'}_{g,i} \quad 1 \le i \le n_g. \end{aligned}$$

It follows that the subspace

$$\left\{ \begin{array}{lll} (v,z) \in \mathcal{V}_2 \times \mathcal{Z}_2 \ : \ z = z[v,z_0] & & \\ \exists \delta > 0 \ \ : \ Dc_i(t,\bar{u}_t,\bar{y}_t)(v_t,z_t) = 0 & t \in \Delta^{\delta}_{c,i} & 1 \le i \le n_c \\ \exists M > 0 \ : \ |Dc_i(t,\bar{u}_t,\bar{y}_t)(v_t,z_t)| \le M & t \in \Delta^{\delta'}_{c,i} & 1 \le i \le n_c \\ \exists \varepsilon > 0 \ \ : \ Dg_i(t,\bar{y}_t)z_t = 0 & t \in \Delta^{\varepsilon}_{g,i} & 1 \le i \le n_g \end{array} \right\}$$

is a dense subset of

$$\left\{ \begin{array}{lll} (v,z) \in \mathcal{V}_2 \times \mathcal{Z}_2 \ : \ z = z[v,z_0] & & \\ Dc_i(t,\bar{u}_t,\bar{y}_t)(v_t,z_t) = 0 & t \in \Delta^0_{c,i} & 1 \le i \le n_c \\ Dg_i(t,\bar{y}_t)z_t = 0 & t \in \Delta^0_{g,i} & 1 \le i \le n_g \end{array} \right\}.$$

Observe now that $C^R_2(\bar{u},\bar{v})$ and $C^S_2(\bar{u},\bar{v})$ are defined by (1.40) and (1.35) respectively as the same polyhedral cone in the previous two vector spaces. See also Remark 1.4.8.2. Then by [38, Lemma 1], the conclusion of Proposition 1.4.19 follows. □

The definition of the set $C^{QR}_2(P^R_N)$ of quasi radial critical directions in L^2 is given in Appendix 1.A.1. Recall that $(\bar{\omega},\bar{\vartheta}) := (0,0) \in \mathcal{A}^N \times \mathbb{R}$.

Lemma 1.4.20. *Let $(v,z) \in C^R_2(\bar{u},\bar{y})$. Then $(v,\bar{\omega},z_0,\bar{\vartheta}) \in C^{QR}_2(P^R_N)$.*

Proof. The direction $(v,\bar{\omega},z_0,\bar{\vartheta})$ is radial [25, Definition 3.52] for the finite dimensional constraints, which are polyhedral, as well as for the constraints on α. Let δ and $M > 0$ be given by definition of $C^R_2(\bar{u},\bar{y})$. Then for any $\sigma > 0$

$$c_i(t,\bar{u}_t,\bar{y}_t) + \sigma Dc_i(t,\bar{u}_t,\bar{y}_t)(v_t,z_t) \le \begin{cases} 0 & \text{for a.a. } t \in \Delta^{\delta}_{c,i} \\ -\delta + \sigma M & \text{for a.a. } t \in \Delta^{\delta'}_{c,i} \setminus \Delta^{\delta}_{c,i} \end{cases}$$

i.e. $(v,\bar{\omega},z_0,\bar{\vartheta})$ is radial for the constraint (1.36). The same argument holds for constraint (1.37) since there exists $\delta_0 > 0$ such that $g_i(t,\bar{y}_t) \le -\delta_0$ for all $t \in \Delta^{\varepsilon'}_{g,i} \setminus \Delta^{\varepsilon}_{g,i}$. Then $(v,\bar{\omega},z_0,\bar{\vartheta})$ is radial, and a fortiori quasi radial. □

Remark 1.4.21. To finish this section, let us mention a flaw in the proof of the density result [22, Lemma 6.4 (ii)]. There is no reason that v^n belongs to L^∞, and not only to L^2, since $(v^n - v)$ is obtained as a preimage of $(w^n - w, \omega^n - \omega)$. The lemma is actually true but its proof requires some effort, see [17, Lemma 4.5] for the case without mixed constraints. The difficulty is avoided here because we do not have to show the density of a L^∞ cone, thanks to our abstract second-order necessary conditions, Theorem 1.A.5, that are derived directly in L^2.

1.A Appendix

1.A.1 Abstract optimization results

In this section, we recall necessary conditions satified by a weak minimum of a general optimal control problem. These conditions have been used in this paper to prove our necessary conditions in Pontryagin form, namely Theorems 1.3.1 and 1.4.9, via the partial relaxation, i.e. Lemmas 1.3.9 and 1.4.16.

We actually state and prove first- and second-order necessary conditions for a more abstract optimization problem. It has to be noted that our second-order conditions, Theorem 1.A.5, are obtained directly on a large set of directions in L^2, thanks to metric regularity result, Lemma 1.A.7, and a tricky truncation, Lemma 1.A.8. To our knowledge, this is new.

Setting

Let K be a nonempty closed convex subset of a Banach space X and $\Delta_1, \dots, \Delta_M$ be measurable sets of $[0,T]$. For $s \in [1,\infty]$, let

$$\begin{aligned} \mathcal{U}_s &:= L^s(0,T;\mathbb{R}^{\bar{m}}), & \mathcal{Y}_s &:= W^{1,s}(0,T;\mathbb{R}^{\bar{n}}), \\ X_s &:= X \times \prod_{i=1}^{M} L^s(\Delta_i), & K_s &:= K \times \prod_{i=1}^{M} L^s(\Delta_i;\mathbb{R}_-). \end{aligned}$$

We consider

$$\begin{aligned} \Gamma \colon \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} &\to \mathcal{Y}_\infty, & J \colon \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} &\to \mathbb{R}, \\ G_1 \colon \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} &\to X, & G_2^i \colon \mathcal{U}_\infty \times \mathcal{Y}_\infty &\to L^\infty(\Delta_i), \end{aligned}$$

the last mappings being defined for $i = 1, \dots, M$ by

$$G_2^i(u,y)_t := m_i(t, u_t, y_t)$$

for a.a. $t \in \Delta_i$, where $m_i \colon [0,T] \times \mathbb{R}^{\bar{m}} \times \mathbb{R}^{\bar{n}} \to \mathbb{R}$. Let

$$G \colon \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}} \to X_\infty, \quad G(u,y^0) := \Big(G_1(u,y^0), G_2(u, \Gamma(u,y^0)) \Big).$$

The optimization problem we consider is the following:

$$\min_{(u,y^0) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}} J(u,y^0) \ ; \ G(u,y^0) \in K_\infty. \tag{AP}$$

Remark 1.A.1. Optimal control problems fit into this framework as follows: given a uniformly quasi-C^1 mapping $F\colon \mathbb{R}\times\mathbb{R}^{\bar m}\times\mathbb{R}^{\bar n}\to\mathbb{R}^{\bar n}$ and the state equation

$$\dot y_t = F(t,u_t,y_t) \quad \text{for a.a. } t\in(0,T), \tag{1.41}$$

we define $\Gamma(u,y^0)$ as the unique $y\in\mathcal{Y}_\infty$ such that (1.41) holds and $y_0=y^0$, for any $(u,y^0)\in\mathcal{U}_\infty\times\mathbb{R}^{\bar n}$; given a cost function $\widetilde J\colon\mathcal{Y}_\infty\to\mathbb{R}$, we define $J:=\widetilde J\circ\Gamma$; given state constraints of any kind (pure, initial-final, ...) $\widetilde G_1\colon\mathcal{Y}_\infty\to X$, with the appropriate space X and convex subset K, we define $G_1:=\widetilde G_1\circ\Gamma$; finally, we define G_2 in order to take into account the mixed control-state and control constraints. By definition, a weak minimum of such an optimal control problem is a locally optimal solution of the corresponding optimization problem (AP).

Assumptions

Let $(\bar u,\bar y^0)$ be feasible for (AP) and let $\bar y:=\Gamma(\bar u,\bar y^0)$. For various Banach spaces Y and mappings $\mathcal{F}\colon\mathcal{U}_\infty\times\mathbb{R}^{\bar n}\to Y$, we will require one of the followings:

Property 1. The mapping $\mathcal{F}$ is C^1 in a neighborhood of $(\bar u,\bar y^0)$, with continuous extensions $D\mathcal{F}(u,y^0)\colon\mathcal{U}_1\times\mathbb{R}^{\bar n}\to Y$.

Property 2. Property 1 holds, and $\mathcal{F}$ is twice Fréchet differentiable at $(\bar u,\bar y^0)$, with a continuous extension $D^2\mathcal{F}(\bar u,\bar y^0)\colon\ \left(\mathcal{U}_2\times\mathbb{R}^{\bar n}\right)^2\to Y$ and the following expansion in Y: for all $(v,z^0)\in\mathcal{U}_\infty\times\mathbb{R}^{\bar n}$,

$$\begin{aligned}\mathcal{F}(\bar u+v,\bar y^0+z^0)=\mathcal{F}(\bar u,\bar y^0)+D\mathcal{F}(\bar u,\bar y^0)(v,z^0)+\frac{1}{2}D^2\mathcal{F}(\bar u,\bar y^0)(v,z^0)^2\\ +o_\infty(\|v\|_2^2+|z^0|^2).\end{aligned}$$

Assumption (i). The mappings Γ, J and G_1 satisfy Property 1, and the functions m_i are uniformly quasi-C^1.

Assumption (i'). The mappings Γ, J and G_1 satisfy Property 2, and the functions m_i are uniformly quasi-C^2.

Assumption (ii). Robinson's constraint qualification holds:

$$0\in\operatorname{int}_{X_\infty}\left\{G(\bar u,\bar y^0)+DG(\bar u,\bar y^0)\left(\mathcal{U}_\infty\times\mathbb{R}^{\bar n}\right)-K_\infty\right\}. \tag{1.42}$$

Assumption (iii). The inward condition holds for G_2: there exists $\gamma>0$ and $\hat v\in\mathcal{U}_\infty$ such that

$$G_2^i(\bar u,\bar y)+D_uG_2^i(\bar u,\bar y)\hat v\le-\gamma \tag{1.43}$$

on Δ_i, $i=1,\ldots,M$.

Remark 1.A.2. Let us consider the case of an optimal control problem, with Γ, J and G_1 defined as in Remark 1.A.1. If F, m_i are uniformly quasi-C^1 and $\widetilde J$, $\widetilde G_1$ are C^1, then Assumption (i) holds. If F, m_i are uniformly quasi-C^2 and $\widetilde J$, $\widetilde G_1$ are C^2, then Assumption (i') holds. See for example [20, Lemmas 19-20] or [85, Theorems 3.3-3.5].

Necessary conditions

We consider the *Lagrangian* $L[\lambda]\colon\mathcal{U}_\infty\times\mathbb{R}^{\bar n}\to\mathbb{R}$, defined for $\lambda\in X_\infty^*$ by

$$L[\lambda](u,y^0):=J(u,y^0)+\left\langle\lambda,G(u,y^0)\right\rangle.$$

We define the set of *Lagrange multipliers* as

$$\Lambda(AP) := \left\{\lambda \in X_1^* \,:\, \lambda \in N_{K_1}\left(G(\bar{u},\bar{y}^0)\right),\, DL[\lambda](\bar{u},\bar{y}^0) = 0 \text{ on } \mathcal{U}_1 \times \mathbb{R}^{\bar{n}}\right\},$$

and the set of *quasi radial critical directions* in L^2 as

$$C_2^{QR}(AP) := \left\{ \begin{array}{c} (v,z^0) \in \mathcal{U}_2 \times \mathbb{R}^{\bar{n}} \,:\, DJ(\bar{u},\bar{y}^0)(v,z^0) \leq 0 \text{ and } \forall \sigma > 0, \\ \operatorname{dist}_{X_1}\left(G(\bar{u},\bar{y}^0) + \sigma DG(\bar{u},\bar{y}^0)(v,z^0), K_1\right) = o(\sigma^2) \end{array} \right\}.$$

We denote by $\operatorname{cl}\left(C_2^{QR}(AP)\right)$ its closure in $\mathcal{U}_2 \times \mathbb{R}^{\bar{n}}$.

Remark 1.A.3. If $(v,z^0) \in C_2^{QR}(AP)$, then $DG(\bar{u},\bar{y}^0)(v,z^0) \in T_{K_1}\left(G(\bar{u},\bar{y}^0)\right)$. If in addition $\Lambda(AP) \neq \emptyset$, then $DJ(\bar{u},\bar{y}^0)(v,z^0) = 0$.

We now state our first- and second-order necessary conditions, in two theorems that will be proved in the next section.

Theorem 1.A.4. *Let $(\bar{u},\bar{y}^0)$ be a locally optimal solution of (AP), and let Assumptions* (i)-(iii) *hold. Then $\Lambda(AP)$ is nonempty, convex, and weakly $*$ compact in X_1^*.*

Theorem 1.A.5. *Let $(\bar{u},\bar{y}^0)$ be a locally optimal solution of (AP), and let Assumptions* (i')-(iii) *hold. Then for any $(v,z^0) \in \operatorname{cl}\left(C_2^{QR}(AP)\right)$, there exists $\lambda \in \Lambda(AP)$ such that*

$$D^2L[\lambda](\bar{u},\bar{y}^0)(v,z^0)^2 \geq 0.$$

Proofs

Proof of Theorem 1.A.4. Robinson's constraint qualification (1.42) and [92, Theorem 4.1] or [25, Theorem 3.9] give the result in X_∞^*. We derive it in X_1^* with the inward condition (1.43), see e.g. [24, Theorem 3.1]. □

Proof of Theorem 1.A.5. (a) Assume first that $(v,z^0) \in C_2^{QR}(AP)$. We consider the following conic linear problem, [25, Section 2.5.6]:

$$\left\{ \begin{array}{l} \displaystyle\min_{(w,\xi^0) \in \mathcal{U}_1 \times \mathbb{R}^{\bar{n}}} DJ(\bar{u},\bar{y}^0)(w,\xi^0) + D^2J(\bar{u},\bar{y}^0)(v,z^0)^2 \;; \\ DG(\bar{u},\bar{y}^0)(w,\xi^0) + D^2G(\bar{u},\bar{y}^0)(v,z^0)^2 \in T_{K_1}\left(G(\bar{u},\bar{y}^0)\right). \end{array} \right. \qquad (Q_{(v,z^0)})$$

Robinson's constraint qualification (1.42) for problem (AP) implies that the constraints of $(Q_{(v,z^0)})$ are regular in the sense of [25, Theorem 2.187]. Then by the same theorem, there is no duality gap between $(Q_{(v,z^0)})$ and its dual, which is the following optimization problem:

$$\max_{\lambda \in \Lambda(AP)} D^2L[\lambda](\bar{u},\bar{y}^0)(v,z^0)^2.$$

Observe indeed that the Lagrangian of $(Q_{(v,z^0)})$ is

$$\mathcal{L}[\lambda](w,\xi^0) = DL[\lambda](\bar{u},\bar{y}^0)(w,\xi^0) + D^2L[\lambda](\bar{u},\bar{y}^0)(v,z^0)^2, \quad \lambda \in X_1^*.$$

The conclusion of the theorem follows when $(v,z^0) \in C_2^{QR}(AP)$ by the following key lemma, that will be proved below.

Lemma 1.A.6. *The value of $(Q_{(v,z^0)})$ is nonnegative.*

(b) Assume now that $(v, z^0) \in \operatorname{cl}(C_2^{QR}(AP))$. Let $(v^k, z^{0,k}) \in C_2^{QR}(AP)$ converge to (v, z^0) in $\mathcal{U}_2 \times \mathbb{R}^{\bar{n}}$. By step (a), there exists $\lambda^k \in \Lambda$ be such that

$$D^2 J(\bar{u}, \bar{y}^0)(v^k, z^{0,k})^2 + \langle \lambda^k, D^2 G(\bar{u}, \bar{y}^0)(v^k, z^{0,k})^2 \rangle = D^2 L[\lambda^k](\bar{u}, \bar{y}^0)(v^k, z^{0,k})^2 \geq 0.$$

By Theorem 1.A.4, there exists $\lambda \in \Lambda$ such that, up to a subsequence, $\lambda^k \rightharpoonup \lambda$ for the weak $*$ topology in X_1^*. By Assumption (i'),

$$D^2 J(\bar{u}, \bar{y}^0) \colon \mathcal{U}_2 \times \mathbb{R}^{\bar{n}} \to \mathbb{R} \quad \text{and} \quad {}^2 G(\bar{u}, \bar{y}^0) \colon \mathcal{U}_2 \times \mathbb{R}^{\bar{n}} \to X_1$$

are continuous. The conclusion follows. $\square$

Proof of Lemma 1.A.6. First we prove a metric regularity result, which relies on Assumption (iii). For any $(u, y) \in \mathcal{U}_\infty \times \mathcal{Y}_\infty$, we define $G_2^+(u, y) \in L^\infty(0, T)$ by

$$G_2^+(u, y)_t := \max_{1 \leq i \leq M} \left(G_2^i(u, y)_t \right)_+$$

for a.a. $t \in (0, T)$, where

$$\left(G_2^i(u, y)_t \right)_+ := \begin{cases} \max\{0, G_2^i(u, y)_t\} & \text{if } t \in \Delta_i, \\ 0 & \text{if } t \notin \Delta_i. \end{cases}$$

Lemma 1.A.7. *There exists $c > 0$ such that, for any $(u, y) \in \mathcal{U}_\infty \times \mathcal{Y}_\infty$ with $y = \Gamma(u, y^0)$ in a neighborhood of $(\bar{u}, \bar{y})$, there exists $(\hat{u}, \hat{y}) \in \mathcal{U}_\infty \times \mathcal{Y}_\infty$ with $\hat{y} = \Gamma(\hat{u}, y^0)$ such that*

$$\|\hat{u} - u\|_\infty \leq c \|G_2^+(u, y)\|_\infty, \tag{1.44}$$
$$\|\hat{u} - u\|_1 \leq c \|G_2^+(u, y)\|_1, \tag{1.45}$$
$$\|G_2^+(\hat{u}, \hat{y})\|_\infty \leq c \|G_2^+(u, y)\|_1. \tag{1.46}$$

Proof. Let $\beta \in (0, 1)$ to be fixed later. Since $(\bar{u}, \bar{y}^0)$ is feasible, $G_2^+(\bar{u}, \bar{y}) = 0$, and there exists $\alpha \in (0, \beta)$ such that

$$\|u - \bar{u}\|_\infty + \|y - \bar{y}\|_\infty \leq \alpha \;\Rightarrow\; \|G_2^+(u, y)\|_\infty \leq \beta.$$

Let (u, y) be such that $\|u - \bar{u}\|_\infty + \|y - \bar{y}\|_\infty \leq \alpha$. We define $\varepsilon \in L^\infty(0, T)$ by

$$\varepsilon_t := \frac{1}{\beta} G_2^+(u, y)_t,$$

so that $\varepsilon_t \in [0, 1]$ for a.a. $t \in (0, T)$, and

$$\hat{u} := u + \varepsilon \hat{v}$$

where $\hat{v}$ is given by the inward condition (1.43). Once β is fixed, it is clear that (1.44) and (1.45) hold. Let $\hat{y} = \Gamma(\hat{u}, y^0)$.

$$\begin{aligned} G_2^i(\hat{u}, \hat{y}) = {} & G_2^i(u, y) + DG_2^i(\bar{u}, \bar{y})(\hat{u} - u, \hat{y} - y) \\ & + \int_0^1 \Big(DG_2^i(u + \theta(\hat{u} - u), y + \theta(\hat{y} - y)) - DG_2^i(\bar{u}, \bar{y}) \Big) (\hat{u} - u, \hat{y} - y) \mathrm{d}\theta \end{aligned}$$

a.e. on Δ_i. Since Γ satisfies Property 1, $\|\hat{y} - y\|_\infty = O\left(\|\hat{u} - u\|_1\right)$, and then

$$\begin{aligned} |\hat{u}_t - u_t| &= O(\varepsilon_t), & |u_t - \bar{u}_t| &= O(\alpha) = O(\beta), \\ |\hat{y}_t - y_t| &= O(\|\varepsilon\|_1), & |y_t - \bar{y}_t| &= O(\alpha) = O(\beta). \end{aligned}$$

Since m_i is uniformly quasi-C^2, G_2^i and DG_2^i are Lipschitz in a neighborhood of $(\bar{u}, \bar{y})$. Then

$$\begin{aligned}G_2^i(\hat{u}, \hat{y}) &= G_2^i(u, y) + \varepsilon D_u G_2^i(\bar{u}, \bar{y})\hat{v} + O\big(\|\varepsilon\|_1 + \varepsilon(\varepsilon + \|\varepsilon\|_1 + \beta)\big) \\ &= (1-\varepsilon)G_2^i(u, y) + \varepsilon\big(G_2^i(u, y) - G_2^i(\bar{u}, \bar{y})\big) \\ &\quad + \varepsilon\big(G_2^i(\bar{u}, \bar{y}) + D_u G_2^i(\bar{u}, \bar{y})\hat{v}\big) + O\big(\|\varepsilon\|_1 + \varepsilon(\varepsilon + \|\varepsilon\|_1 + \beta)\big).\end{aligned}$$

Observe now that

$$\begin{aligned}(1-\varepsilon)G_2^i(u, y) \leq G_2^+(u, y^0) = \varepsilon\beta, \\ \varepsilon\big(G_2^i(u, y) - G_2^i(\bar{u}, \bar{y})\big) = O(\alpha\varepsilon) = O(\varepsilon\beta), \\ \varepsilon\big(G_2^i(\bar{u}, \bar{y}) + D_u G_2^i(\bar{u}, \bar{y})\hat{v}\big) \leq -\varepsilon\gamma.\end{aligned}$$

Then there exists $C > 0$, independent of u and u', such that

$$G_2^i(\hat{u}, \hat{y}) \leq C\|\varepsilon\|_1 + \varepsilon\big[C(\varepsilon + \|\varepsilon\|_1 + \beta) - \gamma\big]$$

on Δ_i, $i = 1, \ldots, M$. We fix $\beta \in (0, 1)$ such that $C\beta \leq \gamma/2$ and $\alpha \in (0, \beta)$ such that $C(\varepsilon + \|\varepsilon\|_1) \leq \gamma/2$. (1.46) follows. □

To prove Lemma 1.A.6, we also need the following:

Lemma 1.A.8. *Let $v \in \mathcal{U}_2$ and $w \in \mathcal{U}_1$. Let $v^k := \mathbf{1}_{\{|v| \leq k\}} v$, $w^k := \mathbf{1}_{\{|w| \leq k\}} w$, and $\sigma_k := \frac{\|v^k - v\|_2}{k}$. Then $v^k, w^k \in \mathcal{U}_\infty$, $\sigma_k \to 0$, and*

$$\|\sigma_k v^k\|_\infty = o(1), \qquad \|\sigma_k^2 w^k\|_\infty = o(1), \tag{1.47}$$

$$\|v^k - v\|_2 = o(1), \qquad \|w^k - w\|_1 = o(1), \tag{1.48}$$

$$\|v^k - v\|_1 = o(\sigma_k). \tag{1.49}$$

Proof. We first get (1.48) by Lebesgue's dominated convergence theorem. Then $\sigma_k = o(\frac{1}{k})$, and (1.47) follows. Observe that $|v^k - v|^2 \geq k|v^k - v|$, which implies

$$\|v^k - v\|_1 = O(\frac{1}{k}\|v^k - v\|_2^2).$$

(1.49) follows by definition of σ_k and by (1.48). □

Let us now go back to the proof of Lemma 1.A.6: let (w, ξ^0) be feasible for problem $(Q_{(v,z^0)})$. We apply Lemma 1.A.8 to $v \in \mathcal{U}_2$, $w \in \mathcal{U}_1$, and we consider

$$\begin{aligned}u^k &:= \bar{u} + \sigma_k v^k + \frac{1}{2}\sigma_k^2 w^k \ \in \mathcal{U}_\infty, \\ y^{0,k} &:= \bar{y}^0 + \sigma_k z^0 + \frac{1}{2}\sigma_k^2 \xi^0 \ \in \mathbb{R}^{\bar{n}}, \\ y^k &:= \Gamma(u^k, y^{0,k}) \ \in \mathcal{Y}_\infty.\end{aligned}$$

We have in particular

$$\|u^k - \bar{u}\|_\infty = o(1), \quad \|u^k - \bar{u}\|_2 = O(\sigma_k). \tag{1.50}$$

By analogy with linearized trajectories, we denote

$$z[\tilde{v}, \tilde{z}^0] := D\Gamma(\bar{u}, \bar{y}^0)(\tilde{v}, \tilde{z}^0), \quad z^2[\tilde{v}, \tilde{z}_0] := D^2\Gamma(\bar{u}, \bar{y}^0)(\tilde{v}, \tilde{z}^0)^2$$

for any $(\tilde{v}, \tilde{z}^0) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}$. Since Γ satisfies Property 2, we have in $\mathcal{Y}_\infty$

$$y^k = \bar{y} + \sigma_k z[v^k, z^0] + \frac{1}{2}\sigma_k^2 \left(z[w^k, \xi^0] + z^2[v^k, z^0] \right) + o(\sigma_k^2), \tag{1.51}$$

and in particular, $\|y^k - \bar{y}\|_\infty = O(\sigma_k)$. Then $(u^k, y^k) \to (\bar{u}, \bar{y})$ in $\mathcal{U}_\infty \times \mathcal{Y}_\infty$ and

$$\|G_2^+(u^k, y^k)\|_\infty = o(1). \tag{1.52}$$

More precisely, since m_i is uniformly quasi-C^2, we have

$$\begin{aligned} G_2^i(u^k, y^k) = G_2^i(\bar{u}, \bar{y}) + DG_2^i(\bar{u}, \bar{y})(u^k - \bar{u}, y^k - \bar{y}) \\ + \frac{1}{2} D^2 G_2^i(\bar{u}, \bar{y})(u^k - \bar{u}, y^k - \bar{y})^2 + o(|u^k - \bar{u}|^2 + |y^k - \bar{y}|^2) \end{aligned}$$

a.e. on Δ_i, where $o(\cdot)$ is uniform w.r.t. t. We write

$$G_2^i(u^k, y^k)_t = \frac{1}{2} T_t^{i,k} + \frac{1}{2} Q_t^{i,k} + R_t^{i,k}$$

where, omitting the time argument t,

$$\begin{aligned} T^{i,k} &:= G_2^i(\bar{u}, \bar{y}) + 2\sigma_k DG_2^i(\bar{u}, \bar{y})(v^k, z[v^k, z^0]), \\ Q^{i,k} &:= G_2^i(\bar{u}, \bar{y}) + \sigma_k^2 (DG_2^i(\bar{u}, \bar{y})(w^k, z[w^k, \xi^0]) \\ &\qquad + D^2 G_2^i(\bar{u}, \bar{y})(v^k, z[v^k, z^0])^2 + D_y G_2^i(\bar{u}, \bar{y}) z^2[v^k, z^0]), \\ R^{i,k} &= \frac{1}{2}\sigma_k^3 D^2 G_2^i(\bar{u}, \bar{y}) \left[(v^k, z[v^k, z^0]), (w^k, z[w^k, \xi^0] + z^2[v^k, z^0] + o(1)) \right] \\ &\quad + \frac{1}{4}\sigma_k^4 D^2 G_2^i(\bar{u}, \bar{y})(w^k, z[w^k, \xi^0] + z^2[v^k, z^0] + o(1))^2 \\ &\qquad + o(|u^k - \bar{u}|^2 + |y^k - \bar{y}|^2) \end{aligned}$$

We claim that $\|R^{i,k}\|_1 = o(\sigma_k^2)$. Indeed, $z[v^k, z^0]$, $z[w^k, \xi^0]$ and $z^2[v^k, z^0]$ are bounded in $\mathcal{Y}_\infty$; the crucial terms are then the following:

$$\begin{aligned} \|\sigma_k^3 D_{uu}^2 G_2^i(\bar{u}, \bar{y})(v^k, w^k)\|_1 &= O\left(\|\sigma_k v^k\|_\infty \cdot \|\sigma_k^2 w^k\|_1 \right) &&= o(\sigma_k^2) \\ \|\sigma_k^4 D_{uu}^2 G_2^i(\bar{u}, \bar{y})(w^k, w^k)\|_1 &= O\left(\|\sigma_k^2 w^k\|_\infty \cdot \|\sigma_k^2 w^k\|_1 \right) &&= o(\sigma_k^2) \\ \|o(|u^k - \bar{u}|^2 + |y^k - \bar{y}|^2)\|_1 &= o\left(\|u^k - \bar{u}\|_2^2 + \|y^k - \bar{y}\|_2^2 \right) &&= o(\sigma_k^2) \end{aligned}$$

by (1.47),(1.48) and (1.50),(1.51). Recall that $(v, z^0) \in C_2^{QR}(AP)$. Then by (1.49) and Property 1, satisfied by Γ, we have

$$\mathrm{dist}_{L^1}\left(T^{i,k}, L^1(\Delta_i; \mathbb{R}_-) \right) = o(\sigma_k^2).$$

Similarly, since (w, ξ^0) is feasible for $(Q_{(v,z^0)})$ and Γ satisfies Property 2,

$$\mathrm{dist}_{L^1}\left(Q^{i,k}, L^1(\Delta_i; \mathbb{R}_-) \right) = o(\sigma_k^2).$$

Then, in addition to (1.52), we have proved that

$$\|G_2^+(u^k, y^k)\|_1 = o(\sigma_k^2).$$

We apply now Lemma 1.A.7 to the sequence (u^k, y^k); we get a sequence $(\hat{u}^k, \hat{y}^k) \in \mathcal{U}_\infty \times \mathcal{Y}_\infty$ with $\hat{y}^k = \Gamma(\hat{u}^k, y^{0,k})$ and such that

$$\begin{aligned} \|\hat{u}^k - u^k\|_\infty &= o(1), \\ \|\hat{u}^k - u^k\|_1 &= o(\sigma_k^2), \\ \|G_2^+(\hat{u}^k, \hat{y}^k)\|_\infty &= o(\sigma_k^2). \end{aligned} \tag{1.53}$$

Since G_1 satisfies Property 2, $(v, z^0) \in C_2^{QR}(AP)$ and (w, ξ^0) is feasible for $(Q_{(v,z^0)})$, we get

$$\mathrm{dist}_X\left(G_1(\hat{u}^k, y^{0,k}), K\right) = o(\sigma_k^2),$$

and then, together with (1.53),

$$\mathrm{dist}_{X_\infty}\left(G(\hat{u}^k, y^{0,k}), K_\infty\right) = o(\sigma_k^2).$$

By Robinson's constraint qualification (1.42), G is metric regular at $(\bar{u}, \bar{y}^0)$ w.r.t. K_∞, [25, Theorem 2.87]. Then there exists $(\tilde{u}^k, \tilde{y}^{0,k}) \in \mathcal{U}_\infty \times \mathbb{R}^{\bar{n}}$ such that

$$\begin{cases} \|\tilde{u}^k - \hat{u}^k\|_\infty + |\tilde{y}^{0,k} - y^{0,k}| = o(\sigma_k^2), \\ G(\tilde{u}^k, \tilde{y}^{0,k}) \in K_\infty. \end{cases}$$

Since $(\bar{u}, \bar{y}^0)$ is a locally optimal solution, $J(\tilde{u}^k, \tilde{y}^{0,k}) \geq J(\bar{u}, \bar{y}^0)$ for k big enough. By Property 2, satisfied by J, we have

$$\sigma_k DJ(\bar{u}, \bar{y}^0)(v, z^0) + \frac{1}{2}\sigma_k^2 (DJ(\bar{u}, \bar{y}^0)(w, \xi^0) + D^2 J(\bar{u}, \bar{y}^0)(v, z^0)^2) + o(\sigma_k^2) \geq 0.$$

The conclusion of Lemma 1.A.6 follows by Theorem 1.A.4 and Remark 1.A.3. □

1.A.2 Proof of Proposition 1.3.8

The proof of Proposition 1.3.8 relies on the following two lemmas, proved at the end of the section. The first one is a consequence of Lyapunov theorem [61] and links relaxed dynamics to classical dynamics.

Lemma 1.A.9. *Let $F\colon [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ and $G\colon [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$ be uniformly quasi-C^1. Let $(\hat{u}, \hat{\alpha}, \hat{y}) \in \mathcal{U} \times \mathcal{A}^N \times \mathcal{Y}$ such that, for a.a. t, $0 \leq \hat{\alpha}^i \leq 1/N$ and*

$$\dot{\hat{y}}_t = \left(1 - \sum_{i=1}^N \hat{\alpha}_t^i\right) F(t, \hat{u}_t, \hat{y}_t) + \sum_{i=1}^N \hat{\alpha}_t^i F(t, u_t^i, \hat{y}_t) + G(t, \hat{y}_t).$$

Then, for any $\varepsilon > 0$, there exists $(u, y) \in \mathcal{U} \times \mathcal{Y}$ such that

$$\dot{y}_t = F(t, u_t, y_t) + G(t, y_t) \quad \text{for a.a. } t, \quad y_0 = \hat{y}_0, \tag{1.54}$$

$$u_t \in \{\hat{u}_t, u_t^1, \ldots, u_t^N\} \quad \text{for a.a. } t, \tag{1.55}$$

$$\|u - \hat{u}\|_1 \leq \sum_{i=1}^N \|\hat{\alpha}^i\|_1 \|u^i - \hat{u}\|_\infty, \tag{1.56}$$

$$\|y - \hat{y}\|_\infty \leq \varepsilon. \tag{1.57}$$

The second one is a metric regularity result, consequence of the qualification of problem (P_N) at $(\bar{u}, \bar{\alpha}, \bar{y}, \bar{\theta})$.

Lemma 1.A.10. *There exists $c > 0$ such that for any relaxed trajectory (u, α, y, θ) with u in a L^1 neighborhood of $\bar{u}$ and (α, y) in a L^∞ neighborhood of $(\bar{\alpha}, \bar{y})$, there exists a relaxed trajectory $(u', \alpha', y', \theta)$ such that*

$$\begin{cases} \|u' - u\|_\infty + \|\alpha' - \alpha\|_\infty + \|y' - y\|_\infty \leq c|\Phi^E(y_0, y_T)|, \\ \Phi^E(y'_0, y'_T) = 0. \end{cases}$$

We can now prove the proposition. The idea is to use alternatively Lemma 1.A.9 to diminish progressively $\hat{\alpha}$, and Lemma 1.A.10 to restore the equality constraints at each step.

Proof of Proposition 1.3.8. Let $(\hat{u}, \hat{y}, \hat{\alpha}, \hat{\theta}) \in F(P_N)$, close to $(\bar{u}, \bar{y}, \bar{\alpha}, \bar{\theta})$ and with $\hat{\theta} < 0$. Without loss of generality, we assume that $\hat{\alpha} \neq 0$ and, see Lemma 1.3.4, that

$$c(t, u_t^i, \hat{y}_t) \leq \hat{\theta} \quad \text{for a.a. } t, \quad 1 \leq i \leq N.$$

Let $R := \operatorname{diam}_{L^\infty} \left\{ \hat{u}, u^1, \dots, u^N \right\}$ and let $\varepsilon > 0$. We claim that there exists a sequence $(\hat{u}^k, \hat{y}^k, \hat{\alpha}^k, \hat{\theta}^k) \in F(P_N)$ such that $(\hat{u}^0, \hat{y}^0, \hat{\alpha}^0, \hat{\theta}^0) = (\hat{u}, \hat{y}, \hat{\alpha}, \hat{\theta})$, and for all k,

$$\operatorname{diam}_{L^\infty} \left\{ \hat{u}^k, u^1, \dots, u^N \right\} < 2R, \tag{1.58}$$

$$c(t, u_t^i, \hat{y}_t^k) \leq \hat{\theta}^k \quad \text{for a.a. } t, \quad 1 \leq i \leq N, \tag{1.59}$$

$$\left\| \hat{u}^{k+1} - \hat{u}^k \right\|_1 \leq \left(\frac{3}{4} \right)^{k+1} 2RNT \|\hat{\alpha}\|_\infty, \tag{1.60}$$

$$\left\| \hat{y}^{k+1} - \hat{y}^k \right\|_\infty \leq \left(\frac{3}{4} \right)^{k+1} \frac{\varepsilon}{4}, \tag{1.61}$$

$$\left\| \hat{\alpha}^{k+1} \right\|_\infty \leq \left(\frac{3}{4} \right)^{k+1} \|\hat{\alpha}\|_\infty, \tag{1.62}$$

$$\hat{\theta}^{k+1} = \frac{1}{4} \hat{\theta}^k. \tag{1.63}$$

Suppose for a while that we have such a sequence. By (1.60)-(1.62), there exist $\tilde{u} \in L^1(0, T; R^m)$ and $\tilde{y} \in C([0, T]; \mathbb{R}^n)$, and $\hat{u}^k \to \tilde{u}$ in L^1, $\hat{y}^k \to \tilde{y}$ in C, and $\hat{\alpha}^k \to 0$ in L^∞. By (1.58), $\tilde{u} \in \mathcal{U}$, and since $(\hat{u}^k, \hat{y}^k, \hat{\alpha}^k, \hat{\theta}^k) \in F(P_N)$ and $\hat{\theta}^k < 0$ for all k, we get that $(\tilde{u}, \tilde{y}) \in F(P)$ by doing $k \to \infty$ in the relaxed dynamics and in the constraints. Finally,

$$\|\tilde{u} - \hat{u}\|_1 \leq 8RNT \|\hat{\alpha} - \bar{\alpha}\|_\infty \quad \text{and} \quad \|\tilde{y} - \hat{y}\|_\infty \leq \varepsilon.$$

It remains to prove the existence the sequence. Suppose we have it up to index k and let us get the next term. Let F^k and G^k be defined by

$$F^k(t, u, y) := \left(1 - \sum_{i=1}^N \frac{\hat{\alpha}_t^{i,k}}{2} \right) f(t, u, y), \quad G^k(t, y) := \sum_{i=1}^N \frac{\hat{\alpha}_t^{i,k}}{2} f(t, u_t^i, y).$$

Since $(\hat{u}^k, \hat{y}^k, \hat{\alpha}^k, \hat{\theta}^k)$ is a relaxed trajectory, we can write

$$\dot{\hat{y}}_t^k = \left(1 - \sum_{i=1}^N \frac{\hat{\alpha}_t^{i,k}/2}{1 - \sum_{j=1}^N \hat{\alpha}_t^{j,k}/2} \right) F^k(t, \hat{u}_t^k, \hat{y}_t^k) + \sum_{i=1}^N \frac{\hat{\alpha}_t^{i,k}/2}{1 - \sum_{j=1}^N \hat{\alpha}_t^{j,k}/2} F^k(t, u_t^i, \hat{y}_t^k) + G^k(t, \hat{y}_t^k).$$

Let $\varepsilon' > 0$. We apply Lemma 1.A.9 and we get $(u, y) \in \mathcal{U} \times \mathcal{Y}$ such that $(u, y, \hat{\alpha}^k/2, \hat{\theta}^k)$ is a relaxed trajectory, and

$$u_t \in \{\hat{u}_t^k, u_t^1, \dots, u_t^N\} \quad \text{for a.a. } t, \tag{1.64}$$

$$\left\| u - \hat{u}^k \right\|_1 \leq \sum_{i=1}^{N} \left\| \frac{\hat{\alpha}_t^{k,i}/2}{1 - \sum \hat{\alpha}_t^{k,j}/2} \right\|_1 \left\| u^i - \hat{u}^k \right\|_\infty, \tag{1.65}$$

$$\left\| y - \hat{y}^k \right\|_\infty \leq \varepsilon'. \tag{1.66}$$

By (1.64), we have

$$\operatorname{diam}_{L^\infty} \left\{ u, u^1, \dots, u^N \right\} \leq \operatorname{diam}_{L^\infty} \left\{ \hat{u}^k, u^1, \dots, u^N \right\} < 2R, \tag{1.67}$$
$$c(t, u_t, \hat{y}_t^k) \leq \hat{\theta}^k \quad \text{for a.a. } t.$$

By (1.66), and since $\hat{\theta}^k < 0$, we have for ε' small enough,

$$\begin{aligned}
&c(t, u_t, y_t) \leq \frac{1}{2}\hat{\theta}^k \quad \text{for a.a. } t, \\
&g(t, y_t) \leq \frac{1}{2}\hat{\theta}^k \quad \text{for a.a. } t, \\
&\Phi^I(y_0, y_T) \leq \frac{1}{2}\hat{\theta}^k, \\
&\phi(y_0, y_T) - \phi(\bar{y}_0, \bar{y}_T) \leq \frac{1}{2}\hat{\theta}^k, \\
&\Phi^E(y_0, y_T) = O(\varepsilon').
\end{aligned} \tag{1.68}$$

Observe that

$$\left| 1 - \sum \hat{\alpha}_t^{k,j}/2 \right| \geq 1 - N\|\hat{\alpha}\|_\infty \geq \frac{3}{4}$$

for $\|\hat{\alpha}\|_\infty$ small enough. Then by (1.62),(1.65) and (1.67),

$$\left\| u - \hat{u}^k \right\|_1 \leq \frac{3}{8} \left(\frac{3}{4} \right)^k 2RNT\|\hat{\alpha}\|_\infty.$$

We now apply Lemma 1.A.10 to $(u, y, \hat{\alpha}^k/2)$ and we get $(\hat{u}^{k+1}, \hat{y}^{k+1}, \hat{\alpha}^{k+1})$ such that $\Phi^E(\hat{y}_0^{k+1}, \hat{y}_T^{k+1}) = 0$ and, by (1.68),

$$\left\| \hat{u}^{k+1} - u \right\|_\infty + \left\| \hat{y}^{k+1} - y \right\|_\infty + \left\| \hat{\alpha}^{k+1} - \frac{\hat{\alpha}^k}{2} \right\|_\infty = O(\varepsilon'). \tag{1.69}$$

Then for $\hat{\theta}^{k+1} := \hat{\theta}^k/4$ and ε' small enough, $(\hat{u}^{k+1}, \hat{y}^{k+1}, \hat{\alpha}^{k+1}, \hat{\theta}^{k+1}) \in F(P_N)$. Moreover,

$$\begin{aligned}
&\operatorname{diam}_{L^\infty} \left\{ \hat{u}^{k+1}, u^1, \dots, u^N \right\} < 2R + \left\| \hat{u}^{k+1} - u \right\|_\infty, \\
&\left\| \hat{u}^{k+1} - \hat{u}^k \right\|_1 \leq \frac{3}{8} \left(\frac{3}{4} \right)^k 2RNT\|\hat{\alpha}\|_\infty + T \left\| \hat{u}^{k+1} - u \right\|_\infty, \\
&\left\| \hat{y}^{k+1} - \hat{y}^k \right\|_\infty \leq \varepsilon' + \left\| \hat{y}^{k+1} - y \right\|_\infty, \\
&\left\| \hat{\alpha}^{k+1} \right\|_\infty \leq \frac{1}{2} \left(\frac{3}{4} \right)^k \|\hat{\alpha}\|_\infty + \left\| \hat{\alpha}^{k+1} - \frac{\hat{\alpha}^k}{2} \right\|_\infty
\end{aligned}$$

By (1.69), and since $\|\hat{\alpha}\|_\infty \neq 0$, we get the sequence up to index $k+1$ for ε' small enough. $\square$

Proof of Lemma 1.A.9. We need the following consequence of Gronwall's lemma:

Lemma 1.A.11. *Let $B\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ be uniformly quasi-C^1. Then there exists $C > 0$ such that, for any $b \in L^\infty(0,T;\mathbb{R}^n)$ and $e^1, e^2 \in \mathcal{Y}$ such that*

$$\begin{cases} \dot{e}^2_t - \dot{e}^1_t = B(t,e^2_t) - B(t,e^1_t) + b_t & \text{for a.a. } t, \\ e^2_0 - e^1_0 = 0, \end{cases}$$

we have

$$\|e^2 - e^1\|_\infty \leq C\|\hat{b}\|_1,$$

where $\hat{b}$ is defined by $\hat{b}_t := \int_0^t b_s \mathrm{d}s$.

Proof. Let $w := e^2 - e^1 - \hat{b}$. Then $\dot{w}_t = B(t,e^2_t) - B(t,e^1_t)$, and

$$|\dot{w}_t| \leq C'|e^2_t - e^1_t| \leq C'(|w_t| + |\hat{b}_t|).$$

The result follows by Gronwall's lemma. $\square$

Let $\varepsilon > 0$, $M \in \mathbb{N}^*$, and $t_j := jT/M$ for $0 \leq j \leq M$. Let us denote by $(e_i)_i$, $1 \leq i \leq N$ the canonical basis of $\mathbb{R}^N$, and let us define $\widetilde{F^i}\colon [0,T] \to \mathbb{R}^n \times \mathbb{R}^N$ by

$$\widetilde{F}^0_t := (F(t,\hat{u}_t,\hat{y}_t),0), \quad \widetilde{F}^i_t := (F(t,u^i_t,\hat{y}_t),e_i) \quad 1 \leq i \leq N.$$

For $0 \leq j < M$, we apply Lyapunov theorem [61] to the family $(\widetilde{F}^i)_i$ with coefficients $(\hat{\alpha}^i)_i$ on $[t_j,t_{j+1}]$. We get the existence of $\alpha \in \mathcal{A}^N$, with values in $\{0,1\}^N$, and such that for $0 \leq j < M$,

$$\int_{t_j}^{t_{j+1}} \Big[\Big(1 - \sum_{i=1}^N \alpha^i_t\Big)\widetilde{F}^0_t + \sum_{i=1}^N \alpha^i_t \widetilde{F}^i_t\Big]\mathrm{d}t = \int_{t_j}^{t_{j+1}} \Big[\Big(1 - \sum_{i=1}^N \hat{\alpha}^i_t\Big)\widetilde{F}^0_t + \sum_{i=1}^N \hat{\alpha}^i_t \widetilde{F}^i_t\Big]\mathrm{d}t. \tag{1.70}$$

Projecting (1.70) on the first n coordinates, we get that

$$\begin{aligned} &\int_{t_j}^{t_{j+1}} \Big[\Big(1 - \sum_{i=1}^N \alpha^i_t\Big)F(t,\hat{u}_t,\hat{y}_t) + \sum_{i=1}^N \alpha^i_t F(t,u^i_t,\hat{y}_t)\Big]\mathrm{d}t \\ &\qquad = \int_{t_j}^{t_{j+1}} \Big[\Big(1 - \sum_{i=1}^N \hat{\alpha}^i_t\Big)F(t,\hat{u}_t,\hat{y}_t) + \sum_{i=1}^N \hat{\alpha}^i_t F(t,u^i_t,\hat{y}_t)\Big]\mathrm{d}t. \end{aligned} \tag{1.71}$$

Let $u_t := \hat{u}_t + \sum_{i=1}^N \alpha^i_t(u^i_t - \hat{u}_t)$. Note that for a.a. t, $u_t \in \{\hat{u}_t,\dots,u^N_t\}$. We get by (1.71) that

$$\int_{t_j}^{t_{j+1}} F(t,u_t,\hat{y}_t)\mathrm{d}t = \int_{t_j}^{t_{j+1}} \Big[\Big(1 - \sum_{i=1}^N \hat{\alpha}^i_t\Big)F(t,\hat{u}_t,\hat{y}_t) + \sum_{i=1}^N \hat{\alpha}^i_t F(t,u^i_t,\hat{y}_t)\Big]\mathrm{d}t. \tag{1.72}$$

Projecting (1.70) on the last N coordinates, we get that for $1 \leq i \leq N$,

$$\int_{t_j}^{t_{j+1}} \alpha^i_t \mathrm{d}t = \int_{t_j}^{t_{j+1}} \hat{\alpha}^i_t \mathrm{d}t. \tag{1.73}$$

Summing (1.73) for $0 \leq j \leq M$, we get that $\|\alpha^i\|_1 = \|\hat{\alpha}^i\|_1$ for $1 \leq i \leq N$. Since

$$\|u - \hat{u}\|_1 \leq \sum_{i=1}^N \|\alpha\|_1 \|u^i - \hat{u}\|_\infty,$$

we get (1.56). Let y be the unique solution of (1.54); we estimate $\|y - \hat{y}\|_\infty$ with Lemma 1.A.11. Let b be defined by

$$b_t := F(t, u_t, \hat{y}_t) - \Big(1 - \sum_{i=1}^{N} \hat{\alpha}_t^i\Big) F(t, \hat{u}_t, \hat{y}_t) - \sum_{i=1}^{N} \hat{\alpha}_t^i F(t, u_t^i, \hat{y}_t),$$

and let $\hat{b}$ be defined by $\hat{b}_t := \int_0^t b_s \mathrm{d}s$. By (1.72), $\hat{b}_{t_j} = 0$ for $0 \leq j \leq M$. Therefore, $\|\hat{b}\|_\infty = O(1/M)$. Observe now that for a.a. t,

$$\dot{y}_t - \dot{\hat{y}}_t = F(t, u_t, y_t) + G(t, y_t) - F(t, u_t, \hat{y}_t) - G(t, \hat{y}_t) + b_t.$$

By Lemma 1.A.11, $\|y - \hat{y}\|_\infty = O(1/M)$. For M large enough, we get (1.57), and the proof is completed. $\square$

Proof of Lemma 1.A.10. Note that the L^1-distance is involved for the control. The lemma is obtained with an extension of the nonlinear open mapping theorem [3, Theorem 5]. This result can be applied since the derivative of the mapping defined in (1.25) can be described explicitely with a linearized state equation and therefore, by Gronwall's lemma, is continuous for the L^1-distance on the control u. $\square$

1.A.3 A qualification condition

Statement

We give here a qualification condition equivalent to the non singularity of generalized Pontryagin multipliers. This qualification condition is expressed with the Pontryagin linearization [69, Proposition 8.1]. In this section, $(\bar{u}, \bar{y}) \in F(P)$ is given. We will always assume that Assumption 2 holds.

Definition 1.A.12. We say that $\lambda = (\beta, \Psi, \nu, \mu) \in \Lambda_L(\bar{u}, \bar{y})$ is *singular* iff $\beta = 0$ and that λ is *normal* iff $\beta = 1$.

Given $u \in \mathcal{U}$, we define the Pontryagin linearization $\xi[u] \in \mathcal{Y}$ as the unique solution of

$$\begin{cases} \dot{\xi}_t[u] = & D_y f(t, \bar{u}_t, \bar{y}_t)\xi_t[u] + f(t, u_t, \bar{y}_t) - f(t, \bar{u}_t, \bar{y}_t), \\ \xi_0[u] = & 0. \end{cases}$$

Note that $\xi[\bar{u}] = 0$. Recall that U is the set-valued mapping defined by (1.8). We define

$$\mathcal{U}_c := \{u \in \mathcal{U} \,:\, u_t \in U(t) \text{ for a.a. } t\}.$$

Definition 1.A.13. We say that the problem is *qualified in the Pontryagin sense* (in short *P-qualified*) at $(\bar{u}, \bar{y})$ iff

(i) the following surjectivity condition holds:

$$0 \in \operatorname{int} \Big\{ D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) \,:\, u \in \mathcal{U}_c, v \in \mathcal{U}, z_0 \in \mathbb{R}^n \Big\}, \tag{1.74}$$

(ii) there exist $\varepsilon > 0$, $\hat{u} \in \mathcal{U}_c$, $\hat{v} \in \mathcal{U}$, and $\hat{z}_0 \in \mathbb{R}^n$ such that

$$D\Phi^E(\bar{y}_0, \bar{y}_T)(\hat{z}_0, \xi_T[\hat{u}] + z_T[\hat{v}, \hat{z}_0]) = 0, \tag{1.75}$$

and for a.a. t,

$$\begin{cases} \Phi^I(\bar{y}_0, \bar{y}_T) + D\Phi^I(\bar{y}_0, \bar{y}_T)(\hat{z}_0, \xi_T[\hat{u}] + z_T[\hat{v}, \hat{z}_0]) \leq -\varepsilon, \\ g(t, \bar{y}_t) + Dg(t, \bar{y}_t)(\xi_t[\hat{u}] + z_t[\hat{v}, \hat{z}_0]) \leq -\varepsilon, \\ c(t, \bar{u}_t, \bar{y}_t) + D_y c(t, \bar{u}_t, \bar{y}_t)\xi_t[\hat{u}] + Dc(t, \bar{u}_t, \bar{y}_t)(\hat{v}_t, z_t[\hat{v}_t, \hat{z}_0]) \leq -\varepsilon. \end{cases} \tag{1.76}$$

Note that if we impose $u = \bar{u}$ in the definition of the P-qualification, we obtain the usual qualification conditions, which are equivalent to the normality of Lagrange multipliers. The P-qualification is then weaker, and as proved in the next theorem, it is necessary and sufficient to ensure the non singularity of Pontryagin multipliers.

Theorem 1.A.14. *Let Assumption 2 hold. Then, the set of singular Pontryagin multipliers is empty if and only if the problem is P-qualified.*

We prove this result in the following two paragraphs.

Proposition 1.A.15. *Let Assumption 2 hold. If the set of singular Pontryagin multipliers is empty, then the set of normal Pontryagin multipliers is bounded in E.*

Proof. Remember that the norm of E is defined by (1.6). We prove the result by contraposition and consider a sequence $(\lambda^k)_k$ of normal Pontryagin multipliers which is such that $\|\lambda^k\|_E \to +\infty$. Then, by Lemma 1.3.5, the sequence $\lambda^k/\|\lambda^k\|_E$ possesses a weak limit point in $\Lambda_P(\bar{u}, \bar{y})$, say $\lambda = (\beta, \Psi, \nu, \mu)$, which is such that

$$\beta = \lim_k \frac{1}{\|\lambda^k\|_E} = 0.$$

Therefore, λ is singular. The proposition is proved. $\square$

Sufficiency of the qualification condition

In this paragraph, we prove by contradiction that the P-qualification implies the non singularity of Pontryagin multipliers. Let us assume that the problem is P-qualified and that there exists $\lambda = (\beta, \Psi, \nu, \mu) \in \Lambda_P(\bar{u}, \bar{y})$ with $\beta = 0$ and $\Psi = (\Psi^E, \Psi^I)$. Let $\hat{u}$, $\hat{w}$, $\hat{z}_0$ be such that (1.75)-(1.76) hold. With an integration by parts and using the stationarity of the augmented Hamiltonian, we get that for all $u \in \mathcal{U}_c$, $v \in \mathcal{U}$, and $z_0 \in \mathbb{R}^n$,

$$\begin{aligned}
&\int_0^T \nu_t (Dc(t, \bar{u}_t, \bar{y}_t)(v_t, z_t[v, z_0]) + D_y c(t, \bar{u}_t, \bar{y}_t)\xi_t[u]) \, \mathrm{d}t \\
&\qquad + \int_0^T Dg(t, \bar{y}_t)(\xi_t[u] + z_t[v, z_0]) \mathrm{d}\mu_t \\
&\qquad + D\Phi[0, (\Psi^E, \Psi^I)](\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) \\
&= \int_0^T H[p_t^\lambda](t, u_t, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) \, \mathrm{d}t \geq 0.
\end{aligned} \tag{1.77}$$

By (1.75)-(1.76) and the nonnegativity of Ψ^I, ν, and μ, we obtain that for $u = \hat{u}$, $v = \hat{v}$, $z_0 = \hat{z}_0$, the r.h.s. of (1.77) is nonpositive and thus equal to 0. Therefore, Ψ^I, ν, and μ are null and for all $u \in \mathcal{U}_c$, $v \in \mathcal{U}$, and $z_0 \in \mathbb{R}^n$,

$$\Psi^E D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) \geq 0. \tag{1.78}$$

By (1.74), we can choose u, v, and z_0 so that for $\beta > 0$ sufficiently small,

$$D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, \xi_T[u] + z_T[v, z_0]) = -\beta(\Psi^E)^T.$$

Combined with (1.78), we obtain that $-\beta|\Psi^E|^2 \geq 0$. Then, $\Psi^E = 0$ and finally $\lambda = 0$, in contradiction with $\lambda \in \Lambda_P(\bar{u}, \bar{y})$.

Necessity of the qualification condition

We now prove that the P-qualification is necessary to ensure the non singularity of Pontryagin multipliers. In some sense, the approach consists in describing this qualification condition as the limit of the qualification conditions associated with a sequence of partially relaxed problems.

Let us fix a Castaing representation $(u^k)_k$ of U. For all $N \in \mathbb{N}$, we consider a partially relaxed problem $(\tilde{P}_N)$ defined by

$$\min_{u\in\mathcal{U},\,\alpha\in\mathcal{A}^N,\,y\in\mathcal{Y}} \phi(y_0, y_T) \quad \text{s.t. constraints (1.2)-(1.4), } y = y[u,\alpha,y_0], \text{ and } \alpha \geq 0, \qquad (\tilde{P}_N)$$

where $y[u,\alpha,y^0]$ is the solution to the partially relaxed state equation (1.17). This problem is the same as problem (P_N), except that there is no variable θ.

For given $v \in \mathcal{U}$, $z_0 \in \mathbb{R}^n$ and $\alpha \in \mathcal{A}^N$, we denote by $z[v,z_0]$ the linearized state variable in the direction (v,z_0), which is the solution to (1.34) and we denote by $\xi[\alpha]$ the linearized state variable in the direction α, which is the solution to

$$\begin{cases} \dot{\xi}_t[\alpha] = & D_y f(t,\bar{u}_t,\bar{y}_t)\xi_t[\alpha] + \sum_{i=1}^N \alpha_t^i\big(f(t,u_t^i,\bar{y}_t) - f(t,\bar{u}_t,\bar{y}_t)\big), \\ \xi_0[\alpha] = & 0. \end{cases}$$

The distinction between the Pontryagin linearization $\xi[u]$ and $\xi[\alpha]$ will be clear in the sequel, and we will motivate this choice of notations in Lemma 1.A.18.

Problem $(\tilde{P}_N)$ is qualified (in the usual sense) iff

(i) the following surjectivity condition holds:

$$0 \in \operatorname{int}\{D\Phi^E(\bar{y}_0,\bar{y}_T)(z_0, \xi_T[\alpha] + z_T[v,z_0]) \,:\, \alpha \in \mathcal{A}^N,\, v \in \mathcal{U},\, z_0 \in \mathbb{R}^n\}$$

(ii) there exist $\varepsilon > 0$, $\hat{\alpha} \in \mathcal{A}^N$, $\hat{v} \in \mathcal{U}$, $\hat{z}_0 \in \mathbb{R}^n$ such that

$$D\Phi^E(\bar{y}_0,\bar{y}_T)(\hat{z}_0, \xi_T[\hat{\alpha}] + z_T[\hat{v},\hat{z}_0]) = 0 \tag{1.79}$$

and

$$\begin{cases} \Phi^I(\bar{y}_0,\bar{y}_T) + D\Phi^I(\bar{y}_0,\bar{y}_T)(\hat{z}_0, \xi_T[\hat{\alpha}] + z_T[\hat{v},\hat{z}_0]) \leq -\varepsilon, \\ g(t,\bar{y}_t) + Dg(t,\bar{y}_t)(\xi_t[\hat{\alpha}] + z_t[\hat{v},\hat{z}_0]) \leq -\varepsilon, \quad \text{for all } t, \\ c(t,\bar{u}_t,\bar{y}_t) + Dc(t,\bar{u}_t,\bar{y}_t)(\hat{v}_t, \xi_t[\hat{\alpha}] + z_t[\hat{v},\hat{z}_0]) \leq -\varepsilon, \quad \text{for a.a. } t, \\ \hat{\alpha}_t \geq \varepsilon, \quad \text{for a.a. } t. \end{cases} \tag{1.80}$$

We denote now by $\Lambda(\tilde{P}_N)$ the set of generalized Lagrange multipliers of problem $(\tilde{P}_N)$ at $(\bar{u}, \alpha = 0, \bar{y})$. Following the proof of Lemma 1.3.10, we easily obtain that

$$\begin{aligned} \Lambda(\tilde{P}_N) = \{(\lambda,\gamma) \in \Lambda^N(\bar{u},\bar{y}) \times L^\infty([0,T];\mathbb{R}^k_+) \,:\, \\ \gamma_t^i = H[p_t^\lambda](t,u_t^i,\bar{y}_t) - H[p_t^\lambda](t,\bar{u}_t,\bar{y}_t), \text{ for } i = 1,...,N, \text{ for a.a. } t\}, \end{aligned} \tag{1.81}$$

where $\Lambda^N(\bar{u},\bar{y})$ is defined by (1.15) and γ is associated with the constraint $\alpha \geq 0$.

Lemma 1.A.16. *Let $N \in \mathbb{N}$; all multipliers of $\Lambda^N(\bar{u},\bar{y})$ are non singular if and only if problem $(\tilde{P}_N)$ is qualified.*

Proof. It is known that all multipliers of $\Lambda(\tilde{P}_N)$ are non singular if and only if problem $(\tilde{P}_N)$ is qualified, see e.g. [25, Proposition 3.16]. It follows from (1.81) that all multipliers of $\Lambda^N(\bar{u},\bar{y})$ are non singular if and only if the multipliers of $\Lambda(\tilde{P}_N)$ are non singular. This proves the lemma. □

As a corollary, we obtain that if problem $(\tilde{P}_N)$ is qualified at stage N, it is also qualified at stage $N+1$. Indeed, if none of the multipliers in $\Lambda^N(\bar{u},\bar{y})$ is singular, *a fortiori*, none of the multipliers in $\Lambda^{N+1}(\bar{u},\bar{y})$ is singular, since $\Lambda^{N+1}(\bar{u},\bar{y}) \subset \Lambda^N(\bar{u},\bar{y})$.

Proposition 1.A.17. *The set of singular Pontryagin multipliers is empty if and only if there exists $N \in \mathbb{N}$ such that problem $(\tilde{P}_N)$ is qualified.*

Proof. Let $N \in \mathbb{N}$ be such that problem $(\tilde{P}_N)$ is qualified. Then, all multipliers of $\Lambda^N(\bar{u},\bar{y})$ are non singular, by Lemma 1.A.16. Since $\Lambda_P(\bar{u},\bar{y}) \subset \Lambda^N(\bar{u},\bar{y})$, the Pontryagin multipliers are non singular.

Conversely, assume that for all N, problem $(\tilde{P}_N)$ is not qualified. By Lemma 1.A.16, we obtain a sequence of singular multipliers $(\lambda^N)_N$ which is such that for all N, $\lambda^N \in \Lambda^N(\bar{u},\bar{y})$. Normalizing this sequence, we obtain with Lemma 1.3.5 the existence of a weak limit point in $\Lambda_P(\bar{u},\bar{y})$, which is necessarily singular. □

To conclude the proof, we still need a relaxation result, which makes a link between the Pontryagin linearization $\xi[u]$ and the linearization $\xi[\alpha]$.

Lemma 1.A.18. *Let $N \in \mathbb{N}$; assume that problem $(\tilde{P}_N)$ is qualified. Then, there exists $A > 0$ such that for all $(\alpha, v, z_0) \in \mathcal{A}^N \times \mathcal{U}, \mathbb{R}^n$ with $\|\alpha\|_\infty \leq A$, $\|v\|_\infty \leq A$, $|z_0| \leq A$, for all $\varepsilon > 0$, if α is uniformly positive, then there exists $(u, v', z_0') \in \mathcal{U}_c \times \mathcal{U} \times \mathbb{R}^n$ such that*

$$\begin{aligned} &D\Phi^E(\bar{y}_0,\bar{y}_T)(z_0, \xi_T[u] + z_T[v,z_0]) = D\Phi^E(\bar{y}_0,\bar{y}_T)(z_0, \xi_T[\alpha] + z_T[v,z_0]), \\ &\|\xi[u] - \xi[\alpha] + z[v'-v, z_0 - z_0']\|_\infty \leq \varepsilon. \end{aligned} \tag{1.82}$$

Proof. We only give some elements of proof. Note that this result is a variant of Proposition 1.3.8 and can be obtained with Dmitruk's result [37, Theorem 3]. Let us define

$$g(t,u,y) := D_y f(t,\bar{u}_t,\bar{y}_t)y + f(t,u,\bar{y}_t) - f(t,\bar{u}_t,\bar{y}_t).$$

Then, for all $u \in \mathcal{U}_c$, $\xi[u]$ is the solution to

$$\dot{\xi}_t[u] = g(t, \xi_t[u], u_t), \quad \xi_0[u] = 0.$$

and $\xi[\alpha]$, where $\alpha \in \mathcal{A}^N$ and $\alpha \geq 0$ is the solution to the relaxed system associated with the dynamics g and the Castaing representation. Indeed,

$$\begin{aligned} \dot{\xi}_t[\alpha] &= D_y f(t,\bar{u}_t,\bar{y}_t)\xi_t[\alpha] + \sum_{i=1}^N \alpha_t^i (f(t,u_t^i,\bar{y}_t) - f(t,\bar{u}_t,\bar{y}_t)) \\ &= \Big(1 - \sum_{i=1}^N \alpha_t^i\Big) g(t,\bar{u}_t,\bar{y}_t) + \sum_{i=1}^N \alpha_t^i (g(t,u_t^i,\bar{y}_t) - g(t,\bar{u}_t,\bar{y}_t)). \end{aligned}$$

Finally, we prove the result by building a sequence $(u^k, \alpha^k, v^k, z_0^k)$ which is such that

$$\begin{aligned} &(u^0,\alpha^0,v^0,z_0^0) = (\bar{u},\alpha,v,z_0), \\ &D\Phi^E(\bar{y}_0,\bar{y}_T)(z_0,\xi_T[\alpha] + z_T[v,z_0]) \\ &\qquad = D\Phi^E(\bar{y}_0,\bar{y}_T)(z_0^k, \xi_T[\alpha^k] + \xi_T[u^k] + z_T[v,z_0]), \end{aligned}$$

such that α^k is uniformly positive and finally which is such that $(u^k)_k$ converges to some $u \in \mathcal{U}_c$ in L^1 norm, $(\alpha^k)_k$ converges to 0 in L^∞ norm, and $(v^k, z_0^k)_k$ equally converges to some (v', z_0') in L^∞ norm. This sequence is built by using Lemma 1.A.9 and by using the surjectivity condition (1.79). Note that Lemma 1.A.9 enables to ensure (1.82). □

Let us conclude the proof of Theorem 1.A.14. Let us assume that the set of singular Pontryagin multipliers is empty; we already know by Proposition 1.A.17 that there exists $N \in \mathbb{N}$ such that the MF_N conditions hold. It remains to prove that the problem is P-qualified. Let $(\alpha^k, v^k, z_0^k)_{k=1,...,n_{\Phi E}+1}$ be such that

$$0 \in \operatorname{int}\Big\{\operatorname{conv}\Big[D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0^k, z_T[v^k, z_0^k] + \xi_T[\alpha^k]),\ k = 1, ..., n_{\phi E} + 1\Big]\Big\}. \tag{1.83}$$

Let $(\hat{\alpha}, \hat{v}, \hat{z}_0)$ be such that (1.80) holds. By (1.79), if we replace (α^k, v^k, z_0^k) by $(\alpha^k+\delta\hat{\alpha}, v^k+\delta\hat{v}, z_0^k + \delta\hat{z}_0)$, for any $\delta > 0$, then (1.83) still holds. Moreover, (1.83) remains true if we multiply this family by a given positive constant. Therefore, since $\hat{\alpha}$ is uniformly positive, we may assume that the family $(\alpha^k, v^k, z_0^k)_{k=1,...,n_{\phi E}+1}$ is bounded by A and such that for all $k = 1, ..., n_{\Phi E}+1$, α^k is uniformly positive. Finally, we can apply Lemma 1.A.18 to any convex combination of elements of the family. This proves the part of the P-qualification associated with equality constraints. Multiplying $(\hat{\alpha}, \hat{v}, \hat{z}_0)$ by a positive constant, we can assume that it is bounded by A and we can equally approximate it so that (1.75) holds and so that (1.76) holds (if the variable ε of Lemma 1.A.18 is chosen sufficiently small). We have proved that the problem was P-qualified.

1.A.4 An example about Pontryagin's principle

We give here an example where there exists a multiplier such that the Hamiltonian inequality (1.9) holds for all $u \in U(t)$, but not for all u in

$$\tilde{U}(t) := \{u \in \mathbb{R}^m \,:\, c(t, u, \bar{y}_t) \le 0\}\,.$$

Indeed, $U(t) \subset \tilde{U}(t)$ but it may happen that $U(t) \neq \tilde{U}(t)$.

Consider the optimal control problem

$$\min y_T$$

subject to the following state equation with fixed initial state, in $\mathbb{R}$:

$$\dot{y}_t = u_t, \quad y_0 = y^0,$$

and to the following mixed constraint:

$$u_t \ge -y_t, \quad \text{for a.a. } t.$$

The optimal control $(\bar{u}, \bar{y})$ is such that $\bar{u}_t = -\bar{y}_t$ and given an initial state y^0, the optimal solution is given by:

$$\bar{u}_t = -y^0 e^{-t}, \quad \bar{y}_t = y^0 e^{-t}.$$

The problem being qualified, there exists a normal Lagrange multiplier which is determined by ν. Since the augmented Hamiltonian is stationary, we obtain that for a.a. t, $p_t^\nu = \nu_t$, and therefore the costate equation writes

$$-\dot{p^\nu}_t = -p_t^\nu, \quad p_T^\nu = 1,$$

i.e. $p_t = \nu_t = e^{-(T-t)} > 0$. Let us fix $y^0 = 0$, the optimal solution is $(0, 0)$ and $\tilde{U}(t) = U(t) = \mathbb{R}_+$. The Hamiltonian pu is minimized for a.a. t by $\bar{u}_t = 0$ since $p_t > 0$.

Now let us consider a variant of this problem. We replace the previous mixed constraint by the following one:

$$\psi(u_t) \ge -y_t,$$

where ψ is a smooth function such that:

$$\begin{cases} \forall u \geq 0, \ \psi(u) = u, \\ \forall u < 0, \ \psi(u) \leq 0 \text{ and } \psi(u) = 0 \Longleftrightarrow u = -1. \end{cases}$$

For $y^0 = 0$, $(0,0)$ remains a feasible trajectory, since $\tilde{U}(t) = \mathbb{R}_+ \cup \{-1\}$. In this case, $U(t) = \mathbb{R}_+$. Let us check that $(0,0)$ is still an optimal solution. Let us suppose that there exist a feasible trajectory (u,y) which is such that $y_T < 0$. Then, let $t \in (0,T)$ be such that

$$y_t \in (y_T, 0) \quad \text{and} \quad \forall s \in [t,T], \ y_s \leq y_t.$$

It follows that for a.a. $s \in (t,T)$,

$$\psi(u_s) \geq -y_s > 0.$$

Therefore, $u_s > 0$ and y is nondecreasing on $[t,T]$, in contradiction with $y_t > y_T$. We have proved that $(0,0)$ is an optimal solution, and the multiplier and costate remain unchanged. However, the minimum of the Hamiltonian over $\tilde{U}(t)$ is reached for

$$u = -1 \neq \bar{u}_t.$$

Chapter 2

Sufficient conditions in Pontryagin form

This chapter is taken from [19]:

J.F. Bonnans, X. Dupuis, L. Pfeiffer. *Second-order sufficient conditions for strong solutions to optimal control problems.* ESAIM Control Optim. Calc. Var., to appear. Inria Research Report No. 8307, May 2013.

In this article, given a reference feasible trajectory of an optimal control problem, we say that the quadratic growth property for bounded strong solutions holds if the cost function of the problem has a quadratic growth over the set of feasible trajectories with a bounded control and with a state variable sufficiently close to the reference state variable. Our sufficient second-order optimality conditions in Pontryagin form ensure this property and ensure *a fortiori* that the reference trajectory is a bounded strong solution. Our proof relies on a decomposition principle, which is a particular second-order expansion of the Lagrangian of the problem.

2.1 Introduction

In this paper, we consider an optimal control problem with final-state constraints, pure state constraints, and mixed control-state constraints. Given a feasible control $\bar{u}$ and its associated state variable $\bar{y}$, we give second-order conditions ensuring that for all $R > \|\bar{u}\|_\infty$, there exist $\varepsilon > 0$ and $\alpha > 0$ such that for all feasible trajectory (u, y) with $\|u\|_\infty \leq R$ and $\|y - \bar{y}\|_\infty \leq \varepsilon$,

$$J(u, y) - J(\bar{u}, \bar{y}) \geq \alpha(\|u - \bar{u}\|_2^2 + |y_0 - \bar{y}_0|^2),$$

where $J(u, y)$ is the cost function to minimize. We call this property *quadratic growth for bounded strong solutions*. Its specificity lies in the fact that the quadratic growth is ensured for controls which may be far from $\bar{u}$ in L^∞ norm.

Our approach is based on the theory of second-order optimality conditions for optimization problems in Banach spaces [34, 58, 66]. A local optimal solution satisfies first- and second-order necessary conditions; denoting by Ω the Hessian of the Lagrangian, theses conditions state that under the extended polyhedricity condition [25, Section 3.2], the supremum of Ω over the set of Lagrange multipliers is nonnegative for all critical directions. Denoting by Ω the Hessian of the Lagrangian, these conditions state that the supremum of Ω over the set of Lagrange multipliers is nonnegative for all critical directions. If the supremum of Ω is positive for nonzero critical directions, we say that the second-order sufficient optimality conditions hold and under some assumptions, a quadratic growth property is then satisfied. This approach can be used for optimal control problems with constraints of any kind. For example, Stefani and Zezza [85] dealt with problems with mixed control-state equality constraints and Bonnans and Hermant [22] with problems with pure state and mixed control-state constraints. However, the quadratic growth property which is then satisfied holds for controls which are sufficiently close to $\bar{u}$ in uniform norm and only ensures that $(\bar{u}, \bar{y})$ is a weak solution.

For Pontryagin minima, that is to say minima locally optimal in a L^1 neighborhood of $\bar{u}$, the necessary conditions can be strengthened. The first-order conditions are nothing but the well-known Pontryagin's principle, historically formulated in [78] and extended to problems with various constraints by many authors, such as Hestenes for problems with mixed control-state constraints [54] Dubovitskii and Osmolovskii for problems with pure state and mixed control-state constraints in early Russian references [40, 41], as highlighted by Dmitruk [36]. We refer to the survey by Hartl *et al.* for more references on this principle.

We say that the second-order necessary condition are in *Pontryagin form* if the supremum of Ω is taken over the set of Pontryagin multipliers, these multipliers being the Lagrange multipliers for which Pontryagin's principle holds. Maurer and Osmolovskii proved in [73] that the second-order necessary conditions in Pontryagin form were satisfied for Pontryagin minima to optimal control problems with mixed control-state equality constraints. They also proved that if second-order sufficient conditions in Pontryagin form held, then the quadratic growth for bounded strong solutions was satisfied. The sufficient conditions in Pontryagin form are as follows: the supremum of Ω over Pontryagin multipliers only is positive for nonzero critical directions and for all bounded neighborhood of $\bar{u}$, there exists a Pontryagin multiplier which is such such the Hamiltonian has itself a quadratic growth. The results of Maurer and Osmolovskii are true under a restrictive full-rank condition for the mixed equality constraints, which is not satisfied by pure constraints, and under the Legendre-Clebsch condition, imposing that the Hessian of the augmented Hamiltonian w.r.t. u is positive. The full-rank condition enabled them to reformulate their their problem as a problem with final-state constraints only. Note that these results were first stated by Milyutin and Osmolovskii in [69], without proof.

For problems with pure and mixed inequality constraints, we proved the second-order necessary conditions in Pontryagin form [18]; in the present paper, we prove that the sufficient conditions in Pontryagin form ensure the quadratic growth property for bounded strong solutions under the Legendre-Clebsch condition. Our proof is based on an extension of the decomposition principle of Bonnans and Osmolovskii [24] to the constrained case. This principle is a particular second-order expansion of the Lagrangian, which takes into account the fact that the control may have large perturbations in uniform norm. Note that the difficulties arising in the extension of the principle and the proof of quadratic growth are mainly due to the presence of mixed control-state constraints.

The outline of the paper is as follows. In Section 2.2, we set our optimal control problem. Section 2.3 is devoted to technical aspects related to the reduction of state constraints. We prove the decomposition principle in Section 2.4 (Theorem 2.4.2) and prove the quadratic growth property for bounded strong solutions in Section 2.5 (Theorem 2.5.3). In Section 2.6, we prove that under technical assumptions, the sufficient conditions are not only sufficient but also necessary to ensure the quadratic growth property (Theorem 2.6.3).

Notations. For a function h that depends only on time t, we denote by h_t its value at time t, by $h_{i,t}$ the value of its i-th component if h is vector-valued, and by $\dot{h}$ its derivative. For a function h that depends on (t, x), we denote by $D_t h$ and $D_x h$ its partial derivatives. We use the symbol D without any subscript for the differentiation w.r.t. all variables except t, e.g. $Dh = D_{(u,y)} h$ for a function h that depends on (t, u, y). We use the same convention for higher order derivatives.

We identify the dual space of $\mathbb{R}^n$ with the space $\mathbb{R}^{n*}$ of n-dimensional horizontal vectors. Generally, we denote by X^* the dual space of a topological vector space X. Given a convex subset K of X and a point x of K, we denote by $T_K(x)$ and $N_K(x)$ the tangent and normal cone to K at x, respectively; see [25, Section 2.2.4] for their definition.

We denote by $|\cdot|$ both the Euclidean norm on finite-dimensional vector spaces and the cardinal of finite sets, and by $\|\cdot\|_s$ and $\|\cdot\|_{q,s}$ the standard norms on the Lesbesgue spaces L^s and the Sobolev spaces $W^{q,s}$, respectively.

We denote by $BV([0, T])$ the space of functions of bounded variation on the closed interval $[0, T]$. Any $h \in BV([0, T])$ has a derivative $\mathrm{d}h$ which is a finite Radon measure on $[0, T]$ and h_0 (resp. h_T) is defined by $h_0 := h_{0_+} - \mathrm{d}h(0)$ (resp. $h_T := h_{T_-} + \mathrm{d}h(T)$). Thus $BV([0, T])$ is endowed with the following norm: $\|h\|_{BV} := \|\mathrm{d}h\|_{\mathcal{M}} + |h_T|$. See [5, Section 3.2] for a rigorous presentation of BV.

All vector-valued inequalities have to be understood coordinate-wise.

2.2 Setting

2.2.1 The optimal control problem

We formulate in this section the optimal control problem under study and we use the same framework as in [18]. We refer to this article for supplementary comments on the different assumptions made. Consider the *state equation*

$$\dot{y}_t = f(t, u_t, y_t) \quad \text{for a.a. } t \in (0, T). \tag{2.1}$$

Here, u is a *control* which belongs to $\mathcal{U}$, y is a *state* which belongs to $\mathcal{Y}$, where

$$\mathcal{U} := L^\infty(0, T; \mathbb{R}^m), \quad \mathcal{Y} := W^{1,\infty}(0, T; \mathbb{R}^n),$$

and $f\colon [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is the *dynamics*. Consider constraints of various types on the system: the *mixed control-state constraints*, or mixed constraints

$$c(t, u_t, y_t) \leq 0 \quad \text{for a.a. } t \in (0,T), \tag{2.2}$$

the *pure state constraints*, or state constraints

$$g(t, y_t) \leq 0 \quad \text{for a.a. } t \in (0,T), \tag{2.3}$$

and the *initial-final state constraints*

$$\begin{cases} \Phi^E(y_0, y_T) = 0, \\ \Phi^I(y_0, y_T) \leq 0. \end{cases} \tag{2.4}$$

Here $c\colon [0,T]\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}^{n_c}$, $g\colon [0,T]\times\mathbb{R}^n \to \mathbb{R}^{n_g}$, $\Phi^E\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n_{\Phi^E}}$, $\Phi^I\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^{n_{\Phi^I}}$. Finally, consider the *cost function* $\phi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The *optimal control problem* is then

$$\min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \phi(y_0, y_T) \quad \text{subject to} \quad (2.1)\text{-}(2.4). \tag{P}$$

We call a *trajectory* any pair $(u,y) \in \mathcal{U}\times\mathcal{Y}$ such that (2.1) holds. We say that a trajectory is *feasible* for problem (P) if it satisfies constraints (2.2)-(2.4), and denote by $F(P)$ the set of feasible trajectories. From now on, we fix a feasible trajectory $(\bar{u}, \bar{y})$.

Similarly to [85, Definition 2.1], we introduce the following Carathéodory-type regularity notion:

Definition 2.2.1. We say that $\varphi\colon [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^s$ is *uniformly quasi-C^k* iff

(i) for a.a. t, $(u,y) \mapsto \varphi(t,u,y)$ is of class C^k, and the modulus of continuity of $(u,y) \mapsto D^k\varphi(t,u,y)$ on any compact of $\mathbb{R}^m \times \mathbb{R}^n$ is uniform w.r.t. t.

(ii) for $j = 0, \dots, k$, for all (u,y), $t \mapsto D^j\varphi(t,u,y)$ is essentially bounded.

Remark 2.2.2. If φ is uniformly quasi-C^k, then $D^j\varphi$ for $j = 0,\dots,k$ are essentially bounded on any compact, and $(u,y) \mapsto D^j\varphi(t,u,y)$ for $j = 0,\dots,k-1$ are locally Lipschitz, uniformly w.r.t. t. In particular, if f is uniformly quasi-C^1, then by Cauchy-Lipschitz theorem, for any $(u, y^0) \in \mathcal{U} \times \mathbb{R}^n$, there exists a unique $y \in \mathcal{Y}$ such that (1.1) holds and $y_0 = y^0$; we denote it by $y[u, y^0]$.

The regularity assumption that we need for the quadratic growth property is the following:

Assumption 1. The mappings f, c and g are uniformly quasi-C^2, g is differentiable, $D_t g$ is uniformly quasi-C^1, Φ^E, Φ^I, and ϕ are C^2.

Note that this assumption will be strengthened in Section 2.6.

Definition 2.2.3. We say that the *inward condition* for the mixed constraints holds iff there exist $\gamma > 0$ and $\bar{v} \in \mathcal{U}$ such that

$$c(t, \bar{u}_t, \bar{y}_t) + D_u c(t, \bar{u}_t, \bar{y}_t)\bar{v}_t \leq -\gamma, \quad \text{for a.a. } t.$$

In the sequel, we will always make the following assumption:

Assumption 2. The inward condition for the mixed constraints holds.

Assumption 2 ensures that the component of the Lagrange multipliers associated with the mixed constraints belongs to $L^\infty(0,T;\mathbb{R}^{n_c*})$, see e.g. [24, Theorem 3.1]. This assumption will also play a role in the decomposition principle.

2.2.2 Bounded strong optimality and quadratic growth

Let us introduce various notions of minima, following [69].

Definition 2.2.4. We say that $(\bar{u}, \bar{y})$ is a *bounded strong minimum* iff for any $R > \|\bar{u}\|_\infty$, there exists $\varepsilon > 0$ such that

$$\begin{aligned} \phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T), \quad & \text{for all } (u, y) \in F(P) \text{ such that} \\ & \|y - \bar{y}\|_\infty \leq \varepsilon \text{ and } \|u\|_\infty \leq R, \end{aligned} \tag{2.5}$$

a *Pontryagin minimum* iff for any $R > \|\bar{u}\|_\infty$, there exists $\varepsilon > 0$ such that

$$\begin{aligned} \phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T), \quad & \text{for all } (u, y) \in F(P) \text{ such that} \\ & \|u - \bar{u}\|_1 + \|y - \bar{y}\|_\infty \leq \varepsilon \text{ and } \|u\|_\infty \leq R, \end{aligned} \tag{2.6}$$

a *weak minimum* iff there exists $\varepsilon > 0$ such that

$$\begin{aligned} \phi(\bar{y}_0, \bar{y}_T) \leq \phi(y_0, y_T), \quad & \text{for all } (u, y) \in F(P) \text{ such that} \\ & \|u - \bar{u}\|_\infty + \|y - \bar{y}\|_\infty \leq \varepsilon. \end{aligned} \tag{2.7}$$

Obviously, (2.5) $\Rightarrow$ (2.6) $\Rightarrow$ (2.7).

Definition 2.2.5. We say that the *quadratic growth property for bounded strong solutions* holds at $(\bar{u}, \bar{y})$ iff for all $R > \|\bar{u}\|_\infty$, there exist $\varepsilon_R > 0$ and $\alpha_R > 0$ such that for all feasible trajectory (u, y) satisfying $\|u\|_\infty \leq R$ and $\|y - \bar{y}\|_\infty \leq \varepsilon$,

$$\phi(y_0, y_T) - \phi(\bar{y}_0, \bar{y}_T) \geq \alpha_R \left(\|u - \bar{u}\|_2^2 + |y_0 - \bar{y}_0|^2 \right).$$

The goal of the article is to characterize this property. If it holds at $(\bar{u}, \bar{y})$, then $(\bar{u}, \bar{y})$ is a bounded strong solution to the problem.

2.2.3 Multipliers

We define the *Hamiltonian* and the *augmented Hamiltonian* respectively by

$$H[p](t, u, y) := pf(t, u, y), \quad H^a[p, \nu](t, u, y) := pf(t, u, y) + \nu c(t, u, y),$$

for $(p, \nu, t, u, y) \in \mathbb{R}^{n*} \times \mathbb{R}^{n_c*} \times [0, T] \times \mathbb{R}^m \times \mathbb{R}^n$. We define the *end points Lagrangian* by

$$\Phi[\beta, \Psi](y_0, y_T) := \beta\phi(y_0, y_T) + \Psi\Phi(y_0, y_T),$$

for $(\beta, \Psi, y_0, y_T) \in \mathbb{R} \times \mathbb{R}^{n_\Phi *} \times \mathbb{R}^n \times \mathbb{R}^n$, where $n_\Phi = n_{\Phi^E} + n_{\Phi^I}$ and $\Phi = \begin{pmatrix} \Phi^E \\ \Phi^I \end{pmatrix}$.

We set

$$K_c := L^\infty(0, T; \mathbb{R}^{n_c}_-), \quad K_g := C([0, T]; \mathbb{R}^{n_g}_-), \quad K_\Phi := \{0\}_{\mathbb{R}^{n_{\Phi^E}}} \times \mathbb{R}^{n_{\Phi^I}}_-,$$

so that the constraints (2.2)-(2.4) can be rewritten as

$$c(\cdot, u, y) \in K_c, \quad g(\cdot, y) \in K_g, \quad \Phi(y_0, y_T) \in K_\Phi.$$

Recall that the dual space of $C([0,T];\mathbb{R}^{n_g})$ is the space $\mathcal{M}([0,T];\mathbb{R}^{n_g*})$ of finite vector-valued Radon measures. We denote by $\mathcal{M}([0,T];\mathbb{R}^{n_g*})_+$ the cone of positive measures in this dual space. Let

$$E := \mathbb{R} \times \mathbb{R}^{n_\Phi *} \times L^\infty(0,T;\mathbb{R}^{n_c*}) \times \mathcal{M}([0,T];\mathbb{R}^{n_g*}).$$

Let $N_{K_c}(c(\cdot,\bar{u},\bar{y}))$ be the set of elements in the normal cone to K_c at $c(\cdot,\bar{u},\bar{y})$ that belong to $L^\infty(0,T;\mathbb{R}^{n_c*})$, i.e.

$$N_{K_c}(c(\cdot,\bar{u},\bar{y})) := \left\{\nu \in L^\infty(0,T;\mathbb{R}_+^{n_c*}) \,:\, \nu_t c(t,\bar{u}_t,\bar{y}_t) = 0 \text{ for a.a. } t\right\}.$$

Let $N_{K_g}(g(\cdot,\bar{y}))$ be the normal cone to K_g at $g(\cdot,\bar{y})$, i.e.

$$N_{K_g}(g(\cdot,\bar{y})) := \left\{\mu \in \mathcal{M}([0,T];\mathbb{R}^{n_g*})_+ \,:\, \int_{[0,T]} (\mathrm{d}\mu_t g(t,\bar{y}_t)) = 0\right\}.$$

Let $N_{K_\Phi}(\Phi(\bar{y}_0,\bar{y}_T))$ be the normal cone to K_Φ at $\Phi(\bar{y}_0,\bar{y}_T)$, i.e.

$$N_{K_\Phi}(\Phi(\bar{y}_0,\bar{y}_T)) := \left\{\Psi \in \mathbb{R}^{n_\Phi *} \,:\, \begin{array}{l}\Psi_i \geq 0\\ \Psi_i\Phi_i(\bar{y}_0,\bar{y}_T) = 0\end{array} \quad \text{for } n_{\Phi^E} < i \leq n_\Phi\right\}.$$

Finally, let

$$N(\bar{u},\bar{y}) := \mathbb{R}_+ \times N_{K_\Phi}(\Phi(\bar{y}_0,\bar{y}_T)) \times N_{K_c}(c(\cdot,\bar{u},\bar{y})) \times N_{K_g}(g(\cdot,\bar{y})) \subset E.$$

We define the *costate space*

$$\mathcal{P} := BV([0,T];\mathbb{R}^{n*}).$$

Given $\lambda = (\beta,\Psi,\nu,\mu) \in E$, we consider the *costate equation* in $\mathcal{P}$

$$\begin{cases} -\mathrm{d}p_t = D_y H^a[p_t,\nu_t](t,\bar{u}_t,\bar{y}_t)\mathrm{d}t + \mathrm{d}\mu_t Dg(t,\bar{y}_t), \\ \;\; p_{T_+} = D_{y_T}\Phi[\beta,\Psi](\bar{y}_0,\bar{y}_T). \end{cases} \tag{2.8}$$

Definition 2.2.6. Let $\lambda = (\beta,\Psi,\nu,\mu) \in E$. We say that the solution of the costate equation (2.8) $p^\lambda \in \mathcal{P}$ is an *associated costate* iff

$$-p^\lambda_{0_-} = D_{y_0}\Phi[\beta,\Psi](\bar{y}_0,\bar{y}_T).$$

Let $N_\pi(\bar{u},\bar{y})$ be the set of *nonzero* $\lambda \in N(\bar{u},\bar{y})$ having an associated costate.

We define the set-valued mapping $U\colon [0,T] \rightrightarrows \mathbb{R}^m$ by

$$U(t) := \mathrm{cl}\left\{u \in \mathbb{R}^m \,:\, c(t,u,\bar{y}_t) < 0\right\} \quad \text{for a.a. } t,$$

where cl denotes the closure in $\mathbb{R}^m$. We can now define two different notions of multipliers.

Definition 2.2.7. (i) We say that $\lambda \in N_\pi(\bar{u},\bar{y})$ is a *generalized Lagrange multiplier* iff

$$D_u H^a[p^\lambda_t,\nu_t](t,\bar{u}_t,\bar{y}_t) = 0 \quad \text{for a.a. } t.$$

We denote by $\Lambda_L(\bar{u},\bar{y})$ the set of generalized Lagrange multipliers.

(ii) We say that $\lambda \in \Lambda_L(\bar{u},\bar{y})$ is a *generalized Pontryagin multiplier* iff

$$H[p^\lambda_t](t,\bar{u}_t,\bar{y}_t) \leq H[p^\lambda_t](t,u,\bar{y}_t) \quad \text{for all } u \in U(t), \quad \text{for a.a. } t. \tag{2.9}$$

We denote by $\Lambda_P(\bar{u},\bar{y})$ the set of generalized Pontryagin multipliers.

Note that the sets $\Lambda_L(\bar{u},\bar{y})$ and $\Lambda_P(\bar{u},\bar{y})$ are convex cones.

Let us mention that we show in [18, Appendix] that even if $(\bar{u},\bar{y})$ is a Pontryagin minimum, inequality (2.9) may not be satisfied for some $t \in [0,T]$ and some $u \in \mathbb{R}^m$ for which $c(t,u,\bar{y}_t) = 0$.

2.2.4 Reduction of touch points

Let us first recall the definition of the order of a state constraint. For $1 \leq i \leq n_g$, assuming that g_i is sufficiently regular, we define by induction $g_i^{(j)} \colon [0,T] \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, $j \in \mathbb{N}$, by

$$g_i^{(j+1)}(t,u,y) := D_t g_i^{(j)}(t,u,y) + D_y g_i^{(j)}(t,u,y) f(t,u,y), \quad g_i^{(0)} := g_i.$$

Definition 2.2.8. If g_i and f are C^{q_i}, we say that the state constraint g_i is of *order* $q_i \in \mathbb{N}$ iff

$$D_u g_i^{(j)} \equiv 0 \quad \text{for } 0 \leq j \leq q_i - 1, \quad D_u g_i^{(q_i)} \not\equiv 0.$$

If g_i is of order q_i, then for all $j < q_i$, $g_i^{(j)}$ is independent of u and we do not mention this dependence anymore. Moreover, the mapping $t \mapsto g_i(t, \bar{y}_t)$ belongs to $W^{q_i,\infty}(0,T)$ and

$$\begin{aligned}
\frac{\mathrm{d}^j}{\mathrm{d}t^j} g_i(t,\bar{y}_t) &= g_i^{(j)}(t,\bar{y}_t) \quad \text{for } 0 \leq j < q_i, \\
\frac{\mathrm{d}^j}{\mathrm{d}t^j} g_i(t,\bar{y}_t) &= g_i^{(j)}(t,\bar{u}_t,\bar{y}_t) \quad \text{for } j = q_i.
\end{aligned}$$

Definition 2.2.9. We say that $\tau \in [0,T]$ is a *touch point* for the constraint g_i iff it is a contact point for g_i, i.e. $g_i(\tau, \bar{y}_\tau) = 0$, and τ is isolated in $\{t : g_i(t,\bar{y}_t) = 0\}$. We say that a touch point τ for g_i is *reducible* iff $\tau \in (0,T)$, $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(t,\bar{y}_t)$ is defined for t close to τ, continuous at τ, and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(t,\bar{y}_t)|_{t=\tau} < 0.$$

For $1 \leq i \leq n_g$, let us define

$$\mathcal{T}_{g,i} := \begin{cases} \emptyset & \text{if } g_i \text{ is of order } 1, \\ \{\text{touch points for } g_i\} & \text{otherwise.} \end{cases}$$

Note that for the moment, we only need to distinguish the constraints of order 1 from the other constraints, for which the order may be undefined if g_i or f is not regular enough.

Assumption 3. For $1 \leq i \leq n_g$, the set $\mathcal{T}_{g,i}$ is finite and only contains reducible touch points.

2.2.5 Tools for the second-order analysis

We define now the linearizations of the system, the critical cone, and the Hessian of the Lagrangian. Let us set

$$\mathcal{V}_2 := L^2(0,T;\mathbb{R}^m), \quad \mathcal{Z}_1 := W^{1,1}(0,T;\mathbb{R}^n), \quad \text{and} \quad \mathcal{Z}_2 := W^{1,2}(0,T;\mathbb{R}^n).$$

Given $v \in \mathcal{V}_2$, we consider the *linearized state equation* in $\mathcal{Z}_2$

$$\dot{z}_t = Df(t,\bar{u}_t,\bar{y}_t)(v_t,z_t) \quad \text{for a.a. } t \in (0,T). \tag{2.10}$$

We call *linerarized trajectory* any $(v,z) \in \mathcal{V}_2 \times \mathcal{Z}_2$ such that (2.10) holds. For any $(v,z^0) \in \mathcal{V}_2 \times \mathbb{R}^n$, there exists a unique $z \in \mathcal{Z}_2$ such that (2.10) holds and $z_0 = z^0$; we denote it by $z = z[v,z^0]$. We also consider the *second-order linearized state equation* in $\mathcal{Z}_1$, defined by

$$\dot{\zeta}_t = D_y f(t,\bar{u}_t,\bar{y}_t)\zeta_t + D^2 f(t,\bar{u}_t,\bar{y}_t)(v_t, z_t[v,z^0])^2 \quad \text{for a.a. } t \in (0,T). \tag{2.11}$$

We denote by $z^2[v, z^0]$ the unique $\zeta \in \mathcal{Z}_1$ such that (2.11) holds and such that $z_0 = 0$.

The critical cone in L^2 is defined by

$$C_2(\bar{u}, \bar{y}) := \left\{ \begin{array}{l} (v, z) \in \mathcal{V}_2 \times \mathcal{Z}_2 \, : \, z = z[v, z_0] \\ D\phi(\bar{y}_0, \bar{y}_T)(z_0, z_T) \leq 0 \\ D\Phi(\bar{y}_0, \bar{y}_T)(z_0, z_T) \in T_{K_\Phi}(\Phi(\bar{y}_0, \bar{y}_T)) \\ Dc(\cdot, \bar{u}, \bar{y})(v, z) \in T_{K_c}(c(\cdot, \bar{u}, \bar{y})) \\ Dg(\cdot, \bar{y})z \in T_{K_g}(g(\cdot, \bar{y})) \end{array} \right\} \tag{2.12}$$

Note that by [25, Examples 2.63 and 2.64], the tangent cones $T_{K_g}(g(\cdot, \bar{y}))$ and $T_{K_c}(c(\cdot, \bar{u}, \bar{y}))$ are resp. described by

$$\begin{aligned} T_{K_g} &= \{\zeta \in C([0, T]; \mathbb{R}^n) \, : \, \forall i, \ \forall t, \ g_i(t, \bar{y}_t) = 0 \Longrightarrow \zeta_{i,t} \leq 0\}, \\ T_{K_c} &= \{w \in L^2([0, T]; \mathbb{R}^m) \, : \, \forall i, \ \text{for a.a. } t, \ c_i(t, \bar{u}_t, \bar{y}_t) = 0 \Longrightarrow w_{i,t} \leq 0\} \end{aligned}$$

Finally, for any $\lambda = (\beta, \Psi, \nu, \mu) \in E$, we define a quadratic form, the *Hessian of Lagrangian*, $\Omega[\lambda]\colon \mathcal{V}_2 \times \mathcal{Z}_2 \to \mathbb{R}$ by

$$\begin{aligned} \Omega[\lambda](v, z) := &\int_0^T D^2 H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)(v_t, z_t)^2 \mathrm{d}t + D^2\Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T)^2 \\ &+ \int_{[0,T]} (\mathrm{d}\mu_t D^2 g(t, \bar{y}_t)(z_t)^2) - \sum_{\substack{\tau \in \mathcal{T}_{g,i} \\ 1 \leq i \leq n_g}} \mu_i(\{\tau\}) \frac{\left(Dg_i^{(1)}(\tau, \bar{y}_\tau) z_\tau\right)^2}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau)}. \end{aligned} \tag{2.13}$$

We justify the terms involving the touch points in $\mathcal{T}_{g,i}$ in the following section.

2.3 Reduction of touch points

We recall in this section the main idea of the reduction technique used for the touch points of state constraints of order greater or equal than 2. Let us mention that this approach was described in [55, Section 3] and used in [63, Section 4] in the case of optimal control problems. As shown in [20], the reduction allows to derive no-gap necessary and sufficient second-order optimality conditions, i.e., the Hessian of the Lagrangian of the reduced problem corresponds to the quadratic form of the necessary conditions. We also prove a strict differentiability property for the mapping associated with the reduction, that will be used in the decomposition principle. Recall that for all $1 \leq i \leq n_g$, all touch points of $\mathcal{T}_{g,i}$ are supposed to be reducible (Assumption 3).

Let $\varepsilon > 0$ be sufficiently small so that for all $1 \leq i \leq n_g$, for all $\tau \in \mathcal{T}_{g,i}$, the time function

$$t \in [\tau - \varepsilon, \tau + \varepsilon] \mapsto g(t, \bar{y}_t)$$

is C^2 and is such that for some $\beta > 0$, $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(t, \bar{y}_t) \leq -\beta$, for all t in $[\tau - \varepsilon, \tau + \varepsilon]$. From now on, we set for all i and for all $\tau \in \mathcal{T}_{g,i}$

$$\Delta_\tau^\varepsilon = [\tau - \varepsilon, \tau + \varepsilon] \quad \text{and} \quad \Delta_i^\varepsilon = [0, T] \backslash \{ \cup_{\tau \in \mathcal{T}_{g,i}} \Delta_\tau^\varepsilon \},$$

and we consider the mapping $\Theta_\tau^\varepsilon : \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Theta_\tau^\varepsilon(u, y^0) := \max \{ g_i(t, y_t) \, : \, y = y[u, y^0], \ t \in \Delta_\tau^\varepsilon \}.$$

We define the reduced pure constraints as follows:

$$\text{for all } i \in \{1, ..., n_g\}, \quad \begin{cases} g_i(t, y_t) \leq 0, & \text{for all } t \in \Delta_i^\varepsilon, & \text{(i)} \\ \Theta_\tau^\varepsilon(u, y_0) \leq 0, & \text{for all } \tau \in \mathcal{T}_{g,i}. & \text{(ii)} \end{cases} \tag{2.14}$$

Finally, we consider the following *reduced problem*, which is an equivalent reformulation of problem (P), in which the pure constraints are replaced by constraint (2.14):

$$\min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \phi(y_0, y_T) \quad \text{subject to} \quad (2.1), (2.2), (2.4), \text{and} (2.14). \tag{P'}$$

Now, for all $1 \leq i \leq n_g$, consider the mapping ρ_i defined by

$$\rho_i : \mu \in \mathcal{M}([0, T]; \mathbb{R}_+) \mapsto \big(\mu_{|\Delta_i^\varepsilon}, (\mu(\tau))_{\tau\in\mathcal{T}_{g,i}}\big) \in \mathcal{M}(\Delta_i^\varepsilon; \mathbb{R}_+) \times \mathbb{R}^{|\mathcal{T}_{g,i}|}.$$

Lemma 2.3.1. *The mapping Θ_τ^ε is twice Fréchet-differentiable at $(\bar{u}, \bar{y}_0)$ with derivatives*

$$\begin{aligned} D\Theta_\tau^\varepsilon(\bar{u}, \bar{y}_0)(v, z_0) &= Dg_i(\tau, \bar{y}_\tau)z_\tau[v, z_0], \\ D^2\Theta_\tau^\varepsilon(\bar{u}, \bar{y}_0)(v, z_0)^2 &= D^2g_i(\tau, \bar{y}_\tau)(z_\tau[v, z_0])^2 + Dg_i(\tau, \bar{y}_\tau)z_\tau^2[v, z_0] \\ &\quad - \frac{\left(Dg_i^{(1)}(\tau, \bar{y}_\tau)z_\tau\right)^2}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau)}. \end{aligned}$$

and the following mappings define a bijection between $\Lambda_L(\bar{u}, \bar{y})$ and the Lagrange multipliers of problem (P'), resp. between $\Lambda_P(\bar{u}, \bar{y})$ and the Pontryagin multipliers of problem (P'):

$$\lambda = (\beta, \Psi, \nu, \mu) \in \Lambda_L(\bar{u}, \bar{y}) \mapsto \big(\beta, \Psi, \nu, (\rho_i(\mu^i))_{1\leq i\leq n_g}\big) \tag{2.15}$$

$$\lambda = (\beta, \Psi, \nu, \mu) \in \Lambda_P(\bar{u}, \bar{y}) \mapsto \big(\beta, \Psi, \nu, (\rho_i(\mu^i))_{1\leq i\leq n_g}\big). \tag{2.16}$$

See [20, Lemma 26] for a proof of this result. Note that the restriction of μ_i to Δ_i^ε is associated with constraint (2.14(i)) and $(\mu_i(\{\tau\}))_{\tau\in\mathcal{T}_{g,i}}$ with constraint (2.14(ii)). The expression of the Hessian of Θ_τ^ε justifies the quadratic form Ω defined in (2.13). Note also that in the sequel, we will work with problem P' and with the original description of the multipliers, using implicitly the bijections (2.15) and (2.16).

Now, let us fix i and $\tau \in \mathcal{T}_{g,i}$. The following lemma is a differentiability property for the mapping Θ_τ^ε, related to the one of strict differentiability, that will be used to prove the decomposition theorem.

Lemma 2.3.2. *There exists $\varepsilon > 0$ such that for all u_1 and u_2 in $\mathcal{U}$, for all y^0 in $\mathbb{R}^n$, if*

$$\|u^1 - \bar{u}\|_1 \leq \varepsilon, \quad \|u^2 - \bar{u}\|_1 \leq \varepsilon, \quad \text{and} \quad |y^0 - \bar{y}_0| \leq \varepsilon, \tag{2.17}$$

then

$$\begin{aligned} \Theta_\tau^\varepsilon(u^2, y^0) - \Theta_\tau^\varepsilon(u^1, y^0) = g(\tau, y_\tau[u^2, y^0]) - g(\tau, y_\tau[u^1, y^0]) \\ + O(\|u^2 - u^1\|_1(\|u^1 - \bar{u}\|_1 + \|u^2 - \bar{u}\|_1 + |y^0 - \bar{y}_0|)). \end{aligned}$$

An intermediate lemma is needed to prove this result. Consider the mapping χ defined as follows:

$$\chi : x \in W^{2,\infty}(\Delta_\tau^\varepsilon) \mapsto \sup_{t\in[\tau-\varepsilon,\tau+\varepsilon]} x_t \in \mathbb{R}.$$

Let us set $x^0 = g_i(\cdot, \bar{y})_{|\Delta_\tau^\varepsilon}$. Note that $\dot{x}_\tau^0 = 0$.

Lemma 2.3.3. *There exists $\alpha' > 0$ such that for all x^1 and x^2 in $W^{2,\infty}(\Delta_\tau)$,if*

$$\|\dot{x}^1 - \dot{x}^0\|_\infty \leq \alpha' \quad \text{and} \quad \|\dot{x}^2 - \dot{x}^0\|_\infty \leq \alpha',$$

then

$$\chi(x^2) - \chi(x^1) = x^2(\tau) - x^1(\tau) + O(\|\dot{x}^2 - \dot{x}^1\|_\infty(\|\dot{x}^1 - \dot{x}^0\|_\infty + \|\dot{x}^2 - \dot{x}^0\|_\infty)). \tag{2.18}$$

Proof. Let $0 < \alpha' < \beta\varepsilon$ and x^1, x^2 in $W^{2,\infty}(\Delta_\tau)$ satisfy the assumption of the lemma. Denote by τ_1 (resp. τ_2) a (possibly non-unique) maximizer of $\chi(x^1)$ (resp. $\chi(x^2)$). Since

$$\dot{x}^1_{\tau-\varepsilon} \geq \dot{x}^0_{\tau-\varepsilon} - \alpha' \geq \beta\varepsilon - \alpha' > 0 \quad \text{and} \quad \dot{x}^1_{\tau+\varepsilon} \leq \dot{x}^0_{\tau+\varepsilon} + \alpha \leq -\beta\varepsilon + \alpha < 0,$$

we obtain that $\tau_1 \in (\tau - \varepsilon, \tau + \varepsilon)$ and therefore that $\dot{x}^1_{\tau_1} = 0$. Therefore,

$$\beta|\tau_1 - \tau| \leq |\dot{x}^0_{\tau_1} - \dot{x}^0_\tau| = |\dot{x}^1_{\tau_1} - \dot{x}^0_{\tau_1}| \leq \|\dot{x}^1 - \dot{x}^0\|_\infty \tag{2.19}$$

and then, $|\tau_1 - \tau| \leq \|\dot{x}^1 - \dot{x}^0\|_\infty/\beta$. Similarly, $|\tau_2 - \tau| \leq \|\dot{x}^2 - \dot{x}^0\|_\infty/\beta$. Then, by (2.19),

$$\begin{aligned}\chi(x^2) &\geq x^1(\tau_1) + (x^2(\tau_1) - x^1(\tau_1)) \\ &= \chi(x^1) + (x^2(\tau) - x^1(\tau)) + O(\|\dot{x}^2 - \dot{x}^1\|_\infty|\tau_1 - \tau|)\end{aligned}$$

and therefore, the l.h.s. of (2.18) is greater than the r.h.s. and by symmetry, the converse inequality holds. The lemma is proved. □

Proof of Lemma 2.3.2. Consider the mapping

$$G_\tau : (u, y^0) \in (\mathcal{U} \times \mathbb{R}^n) \mapsto (t \in \Delta_\tau \mapsto g_i(t, y_t[u, y^0])) \in W^{2,\infty}(\Delta_\tau).$$

Since g_i is not of order 1 and by Assumption 1, the mapping G_τ is Lipschitz in the following sense : there exists $K > 0$ such that for all $(u^1, y^{0,1})$ and $(u^2, y^{0,2})$,

$$\|G_\tau(u^1, y^{0,1}) - G_\tau(u^2, y^{0,2})\|_{1,\infty} \leq K(\|u^2 - u^1\|_1 + |y^{0,2} - y^{0,1}|). \tag{2.20}$$

Set $\alpha = \alpha'/(2K)$. Let u^1 and u^2 in $\mathcal{U}$, let y^0 in $\mathbb{R}^n$ be such that (2.17) holds. Then by Lemma 2.3.3 and by (2.20),

$$\begin{aligned}\Theta^\varepsilon_\tau(u^2, y^0) - \Theta^\varepsilon_\tau(u^1, y^0) &= \chi(G_\tau(u^2, y^0)) - \chi(G_\tau(u^1, y^0)) \\ &= g(y_\tau[u^2, y^0]) - g(y_\tau[u^1, y^0]) \\ &\qquad + O(\|u^2 - u^1\|_1(\|u^2 - \bar{u}\|_1 + \|u^1 - \bar{u}\|_1 + |y^0 - \bar{y}_0|)),\end{aligned}$$

as was to be proved. □

2.4 A decomposition principle

We follow a classical approach by contradiction to prove the quadratic growth property for bounded strong solutions. We assume the existence of a sequence of feasible trajectories $(u^k, y^k)_k$ which is such that u^k is bounded and such that $\|y^k - \bar{y}\|_\infty \to 0$ and for which the quadratic growth property does not hold. The Lagrangian function first provides a lower estimate of the cost function $\phi(y^k_0, y^k_T)$. The difficulty here is to linearize the Lagrangian, since we must consider large perturbations of the control in L^∞ norm. To that purpose, we extend the decomposition principle of [24, Section 2.4] to our more general framework with pure and mixed constraints. This principle is a partial expansion of the Lagrangian, which is decomposed into two terms: $\Omega[\lambda](v^{A,k}, z[v^{A,k}, y^k_0 - \bar{y}_0])$, where $v^{A,k}$ stands for the small perturbations of the optimal control, and a difference of Hamiltonians where the large perturbations occur.

2.4.1 Notations and first estimates

Let $R > \|\bar{u}\|_\infty$, let $(u^k, y^k)_k$ be a sequence of feasible trajectories such that

$$\forall k,\ \|u^k\|_\infty \leq R \quad \text{and} \quad \|u^k - \bar{u}\|_2 + |y_0^k - \bar{y}_0| \to 0. \tag{2.21}$$

This sequence will appear in the proof of the quadratic growth property. Note that the convergence of controls and initial values of state implies that $\|y^k - \bar{y}\|_\infty \to 0$. We need to build two auxiliary controls $\tilde{u}^k$ and $u^{A,k}$. The first one, $\tilde{u}^k$, is such that

$$\begin{cases} c(t, \tilde{u}_t^k, y_t^k) \leq 0, \text{ for a.a. } t \in [0,T], \\ \|\tilde{u}^k - \bar{u}\|_\infty = O(\|y^k - \bar{y}\|_\infty). \end{cases} \tag{2.22}$$

The following lemma proves the existence of such a control.

Lemma 2.4.1. *There exist $\varepsilon > 0$ and $\alpha \geq 0$ such that for all $y \in \mathcal{Y}$ with $\|y - \bar{y}\|_\infty \leq \varepsilon$, there exists $u \in \mathcal{U}$ satisfying*

$$\|u - \bar{u}\|_\infty \leq \alpha \|y - \bar{y}\|_\infty \quad \textit{and} \quad c(t, u_t, y_t) \leq 0, \textit{ for a.a. } t.$$

Proof. For all $y \in \mathcal{Y}$, consider the mapping C_y defined by

$$u \in \mathcal{U} \mapsto C_y(u) = \big(t \mapsto c(t, u_t, y_t)\big) \in L^\infty(0,T;\mathbb{R}^{n_g}).$$

The inward condition (Assumption 2) corresponds to Robinson's constraint qualification for $C_{\bar{y}}$ at $\bar{u}$ with respect to $L^\infty(0,T;\mathbb{R}_-^{n_g})$. Thus, by the Robinson-Ursescu stability theorem [25, Theorem 2.87], there exists $\varepsilon > 0$ such that for all $y \in \mathcal{Y}$ with $\|y - \bar{y}\|_\infty \leq \varepsilon$, C_y is metric regular at $\bar{u}$ with respect to $L^\infty(0,T;\mathbb{R}_-^{n_g})$. Therefore, for all $y \in \mathcal{Y}$ with $\|y - \bar{y}\|_\infty \leq \varepsilon$, there exists a control u such that, for almost all t, $c(t, u_t, y_t) \leq 0$ and

$$\|u - \bar{u}\|_\infty = O\big(\text{dist}(C_y(\bar{u}), L^\infty(0,T;\mathbb{R}_-^{n_g}))\big) = O(\|y - \bar{y}\|_\infty).$$

This proves the lemma. □

Now, let us introduce the second auxiliary control $u^{A,k}$. We say that a partition (A,B) of the interval $[0,T]$ is measurable iff A and B are measurable subset of $[0,T]$. Let us consider a sequence of measurable partitions $(A_k, B_k)_k$ of $[0,T]$. We define $u^{A,k}$ as follows:

$$u_t^{A,k} = \bar{u}_t \mathbf{1}_{\{t \in B^k\}} + u_t^k \mathbf{1}_{\{t \in A_k\}}. \tag{2.23}$$

The idea is to separate, in the perturbation $u^k - \bar{u}$, the small and large perturbations in uniform norm. In the sequel, the letter A will refer to the small perturbations and the letter B to the large ones. The large perturbations will occur on the subset B_k.

For the sake of clarity, we suppose from now that the following holds:

$$\begin{cases} (A_k, B_k)_k \text{ is a sequence of measurable partitions of } [0,T], \\ |y_0^k - \bar{y}_0| + \|u^{A,k} - \bar{u}\|_\infty \to 0, \\ |B_k| \to 0, \end{cases} \tag{2.24}$$

where $|B_k|$ is the Lebesgue measure of B_k. We set

$$v^{A,k} := u^{A,k} - \bar{u} \quad \text{and} \quad v^{B,k} := u^k - u^{A,k}$$

and we define

$$\delta y^k := y^k - \bar{y}, \quad y^{A,k} := y[u^{A,k}, y_0^k], \quad \text{and} \quad z^{A,k} := z[v^{A,k}, \delta y_0^k].$$

Let us introduce some useful notations for the future estimates:

$$\begin{aligned} R_{1,k} &:= \|u^k - \bar{u}\|_1 + |\delta y_0^k|, & R_{2,k} &:= \|u^k - \bar{u}\|_2 + |\delta y_0^k|, \\ R_{1,A,k} &:= \|v^{A,k}\|_1 + |\delta y_0^k|, & R_{2,A,k} &:= \|v^{A,k}\|_2 + |\delta y_0^k|, \\ R_{1,B,k} &:= \|v^{B,k}\|_1, & R_{2,B,k} &:= \|v^{B,k}\|_2. \end{aligned} \tag{2.25}$$

Combining the Cauchy-Schwarz inequality and assumption (2.24), we obtain that

$$R_{1,B,k} \leq R_{2,B,k}|B_k|^{1/2} = o(R_{2,B,k}).$$

Note that by Gronwall's lemma,

$$\|\delta y^k\|_\infty = O(R_{1,k}) = O(R_{2,k}) \quad \text{and} \quad \|z^{A,k}\|_\infty = O(R_{1,A,k}) = O(R_{2,k}).$$

Note also that

$$\|\delta y^k - (y^{A,k} - \bar{y})\|_\infty = O(R_{1,B,k}) = o(R_{2,k})$$

and since $\|y^{A,k} - (\bar{y} + z^{A,k})\|_\infty = O(R_{2,k}^2)$,

$$\|\delta y^k - z^{A,k}\|_\infty = o(R_{2,k}). \tag{2.26}$$

2.4.2 Result

We can now state the decomposition principle.

Theorem 2.4.2. *Suppose that Assumptions 1, 2, and 3 hold. Let $R > \|\bar{u}\|_\infty$, let $(u^k, y^k)_k$ be a sequence of feasible controls satisfying* (2.21) *and $(A_k, B_k)_k$ satisfy* (2.24). *Then, for all $\lambda = (\beta, \Psi, \nu, \mu) \in \Lambda_L(\bar{u}, \bar{y})$,*

$$\begin{aligned} \beta(\phi(y_0^k, y_T^k) - \phi(\bar{y}_0, \bar{y}_T)) \geq{}& \tfrac{1}{2}\Omega[\lambda](v^{A,k}, z^{A,k}) \\ &+ \int_{B_k} [H[p_t^\lambda](t, u_t^k, \bar{y}_t) - H[p_t^\lambda](t, \tilde{u}_t^k, \bar{y}_t)]\mathrm{d}t + o(R_{2,k}^2), \end{aligned} \tag{2.27}$$

where Ω is defined by (2.13).

The proof is given at the end of the section. The basic idea to obtain a lower estimate of $\beta(\phi(y_0, y_T) - \phi(\bar{y}_0, \bar{y}_T))$ is classical: we dualize the constraints and expand up to the second order the obtained Lagrangian. However, the dualization of the mixed constraint is particular here, in so far as the nonpositive added term is the following:

$$\int_{A_k} \nu_t(c(t, u_t^{A,k}, y_t^k) - c(t, \bar{u}_t, \bar{y}_t))\,\mathrm{d}t + \int_{B_k} \nu_t(c(t, \tilde{u}_t^k, y_t^k) - c(t, \bar{u}_t, \bar{y}_t))\,\mathrm{d}t,$$

where $\tilde{u}^k$ and $u^{A,k}$ are defined by (2.22) and (2.23). In some sense, we do not dualize the mixed constraint when there are large perturbations of the control. By doing so, we prove that the contribution of the large perturbations is of the same order as the difference of Hamiltonians appearing in (2.27). If we dualized the mixed constraint with the following term:

$$\int_0^T \nu_t(c(t, u_t^k, y_t^k) - c(t, \bar{u}_t, \bar{y}_t))\,\mathrm{d}t,$$

we would obtain for the contribution of large perturbations a difference of augmented Hamiltonians.

Let us fix $\lambda \in \Lambda_L(\bar{u}, \bar{y})$ and let us consider the following two terms:

$$
\begin{aligned}
I_1^k = \int_0^T & -H_y^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)\delta y_t^k \,\mathrm{d}t \\
& + \int_{A_k} (H^a[p_t^\lambda, \nu_t](t, u_t^{A,k}, y_t^k) - H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)) \,\mathrm{d}t && (2.28a)\\
& + \int_{B_k} (H^a[p_t^\lambda, \nu_t](t, \tilde{u}_t^k, y_t^k) - H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)) \,\mathrm{d}t && (2.28b)\\
& + \int_{B_k} (H[p_t^\lambda](t, u_t^k, y_t^k) - H[p_t^\lambda](t, \tilde{u}_t^k, y_t^k)) \,\mathrm{d}t && (2.28c)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2^k = & -\int_{[0,T]} (\mathrm{d}\mu_t Dg(t, \bar{y}_t)\delta y_t^k) + \sum_{i=1}^{n_g} \int_{\Delta_i^\varepsilon} (g_i(t, y_t^k) - g_i(t, \bar{y}_t)) \,\mathrm{d}\mu_{t,i} && (2.29a)\\
& + \sum_{\substack{\tau \in \mathcal{T}_{g,i} \\ 1 \leq i \leq n_g}} \mu_i(\tau)(\Theta_\tau^\varepsilon(u^k, y_0^k) - \Theta_\tau^\varepsilon(\bar{u}, \bar{y}_0)). && (2.29b)
\end{aligned}
$$

Lemma 2.4.3. *Let $R > \|\bar{u}\|_\infty$, let $(u^k, y^k)_k$ be a sequence of feasible trajectories satisfying (2.21), and let $(A_k, B_k)_k$ satisfy (2.24). Then, for all $\lambda \in \Lambda_L(\bar{u}, \bar{y})$, the following lower estimate holds:*

$$
\beta(\phi(y_0^k, y_T^k) - \phi(\bar{y}_0, \bar{y}_T)) \geq \tfrac{1}{2} D^2\Phi[\beta, \lambda](\bar{y}_0, \bar{y}_T)(z_0^{A,k}, z_T^{A,k})^2 + I_1^k + I_2^k + o(R_{2,k}^2).
$$

Proof. Let $\lambda \in \Lambda_L(\bar{u}, \bar{y})$. In view of sign conditions for constraints and multipliers, we first obtain that

$$
\begin{aligned}
\beta\phi(y_0^k, y_T^k) - \phi(\bar{y}_0, \bar{y}_T) \geq & \; \Phi[\beta, \Psi](y_0^k, y_T^k) - \Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T) \\
& + \sum_{i=1}^{n_g} \int_{\Delta_i^\varepsilon} (g_i(t, y_t^k) - g_i(t, \bar{y}_t)) \,\mathrm{d}\mu_{i,t} + \sum_{\substack{\tau \in \mathcal{T}_{g,i} \\ 1 \leq i \leq n_g}} \mu_i(\tau)(\Theta_\tau^\varepsilon(u^k, y_0^k) - \Theta_\tau^\varepsilon(\bar{u}, \bar{y}_0)) \\
& + \int_{A_k} \nu_t(c(t, u_t^{A,k}, y_t^k) - c(t, \bar{u}_t, \bar{y}_t)) \,\mathrm{d}t + \int_{B_k} \nu_t(c(t, \tilde{u}_t^k, y_t^k) - c(t, \bar{u}_t, \bar{y}_t)) \,\mathrm{d}t. \quad (2.30)
\end{aligned}
$$

Expanding the end-point Lagrangian up to the second order, and using (2.26), we obtain that

$$
\begin{aligned}
& \Phi[\beta, \Psi](y_0^k, y_T^k) - \Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T) \\
= & \; D\Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T)(\delta y_0^k, \delta y_T^k) + \tfrac{1}{2} D^2\Phi[\beta, \Psi](\bar{y}_0, \bar{y}_T)(\delta y_0^k, \delta y_T^k)^2 + o(R_{2,k}^2) \\
= & \; (p_T^\lambda \delta y_T^k - p_0^\lambda \delta y_0^k) + \tfrac{1}{2} D^2\Phi[\lambda](\bar{y}_0, \bar{y}_T)(z_0^{A,k}, z_T^{A,k})^2 + o(R_{2,k}^2). \quad (2.31)
\end{aligned}
$$

Integrating by parts (see [20, Lemma 32]), we obtain that

$$
\begin{aligned}
p_T^\lambda \delta y_T^k - p_0^\lambda \delta y_0^k = & \int_{[0,T]} (\mathrm{d}p_t^\lambda \delta y_t^k + p_t^\lambda \delta \dot{y}_t^k \,\mathrm{d}t) \\
= & \int_0^T \big(- H_y^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)\delta y_t^k + H[p_t^\lambda](t, u_t^k, y_t^k) - H[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)\big)\mathrm{d}t \\
& - \int_{[0,T]} (\mathrm{d}\mu_t Dg(t, \bar{y}_t)\delta y_t^k). \quad (2.32)
\end{aligned}
$$

The lemma follows from (2.30), (2.31), and (2.32). □

A corollary of Lemma 2.4.3 is the following estimate, obtained with (2.22):

$$\begin{aligned}
&\beta(\phi(y_0^k, y_T^k) - \phi(\bar{y}_0, \bar{y}_T)) \\
&\geq \int_0^T [H[p_t^\lambda](t, u_t^k, y_t^k) - H[p_t^\lambda](t, \tilde{u}_t^k, y_t^k)]\,\mathrm{d}t + O(\|\delta y^k\|_\infty) \\
&= \int_0^T [H[p_t^\lambda](t, u_t^k, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t)]\,\mathrm{d}t + O(\|\delta y^k\|_\infty).
\end{aligned} \tag{2.33}$$

Proof of the decomposition principle. We prove Theorem 2.4.2 by estimating the terms I_1^k and I_2^k obtained in Lemma 2.4.3.

$\triangleright$ *Estimation of I_1^k.*

Let show that

$$\begin{aligned}
I_1^k = \frac{1}{2} \int_0^T D^2 H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)(v_t^{A,k}, z_t^{A,k})^2\,\mathrm{d}t \\
+ \int_{B_k} (H[p_t^\lambda](t, u_t^k, \bar{y}_t) - H[p_t^\lambda](t, \tilde{u}_t^k, \bar{y}_t))\,\mathrm{d}t + o(R_{2,k}^2).
\end{aligned} \tag{2.34}$$

Using (2.26) and the stationarity of the augmented Hamiltonian, we obtain that term (2.28a) is equal to

$$\int_{A_k} H_y^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)\delta y_t^k \mathrm{d}t + \frac{1}{2} \int_{A_k} D^2 H^a[p_t^\lambda, \nu_t](t, \bar{u}_t, \bar{y}_t)(v_t^{A,k}, z_t^{A,k})^2 \mathrm{d}t + o(R_{2,k}^2). \tag{2.35}$$

Term (2.28b) is negligible compared to $R_{2,k}^2$. Since

$$\begin{aligned}
\int_{B_k} (H[p_t^\lambda](t, u_t^k, y_t^k) - H[p_t^\lambda](t, \tilde{u}_t^k, y_t^k))\,\mathrm{d}t - \int_{B_k} (H[p_t^\lambda](t, u_t^k, \bar{y}_t) - H[p_t^\lambda](t, \tilde{u}_t^k, \bar{y}_t))\,\mathrm{d}t \\
= O(|B_k| R_{1,k}^2) = o(R_{2,k}^2),
\end{aligned} \tag{2.36}$$

term (2.28c) is equal to

$$\int_{B_k} (H[p_t^\lambda](t, u_t^k, \bar{y}_t) - H[p_t^\lambda](t, \tilde{u}_t^k, \bar{y}_t))\,\mathrm{d}t + o(R_{2,k}^2). \tag{2.37}$$

The following term is also negligible:

$$\int_{B_k} D^2 H^a[p_t^\lambda](t, \bar{u}_t, \bar{y}_t)(v_t^{A,k}, z_t^{A,k})^2 \mathrm{d}t = o(R_{2,k}^2). \tag{2.38}$$

Finally, combining (2.28), (2.35), (2.37), and (2.38), we obtain (2.34).

$\triangleright$ *Estimation of I_2^k.*

Let us show that

$$I_2^k = \frac{1}{2} \int_{[0,T]} (\mathrm{d}\mu_t D^2 g(t, \bar{y}_t)(z_t^{A,k})^2) - \frac{1}{2} \sum_{\substack{\tau \in \mathcal{T}_{g,i} \\ 1 \leq i \leq n_g}} \mu_i(\tau) \frac{(Dg_i^{(1)}(\tau, \bar{y}_\tau) z_\tau^{A,k})^2}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau)}. \tag{2.39}$$

Using (2.26), we obtain the following estimate of term (2.29a):

$$- \sum_{\substack{\tau \in \mathcal{T}_{g,i} \\ 1 \leq i \leq n_g}} \int_{\Delta_\tau^\varepsilon} Dg_i(t, \bar{y}_t) \delta y_t^k \mathrm{d}\mu_{i,t} + \frac{1}{2} \sum_{i=1}^{n_g} \int_{\Delta_i^\varepsilon} D^2 g_i(t, \bar{y}_t)(z_\tau^{A,k})^2 \mathrm{d}\mu_t + o(R_{2,k}^2). \tag{2.40}$$

Remember that $z^2[v^{A,k}, \delta y_0^k]$ denotes the second-order linearization (2.11) and that the following holds:

$$\|y^{A,k} - (\bar{y} + z[v^{A,k}, \delta y_0^k] + z^2[v^{A,k}, \delta y_0^k])\|_\infty = o(R_{2,k}^2).$$

Using Lemma 2.3.2 and estimate (2.26), we obtain that for all i, for all $\tau \in \mathcal{T}_{g,i}$,

$$\begin{aligned} \Theta_\tau^\varepsilon(u^k, y_0^k) - \Theta_\tau^\varepsilon(u^{A,k}, y_0^k) &= g_i(\tau, y_\tau^k) - g_i(\tau, y_\tau^{A,k}) + O(R_{1,B,k}(R_{1,B,k} + R_{1,k})) \\ &= Dg_i(\tau, \bar{y}_\tau)(y_\tau^k - y_\tau^{A,k}) + o(R_{2,k}^2) \\ &= Dg_i(\tau, \bar{y}_\tau)(\delta y_\tau^k - z_\tau^{A,k} - z_\tau^2[v^{A,k}, \delta y_0^k]) + o(R_{2,k}^2). \end{aligned} \tag{2.41}$$

By Lemma 2.3.1,

$$\begin{aligned} \Theta_\tau^\varepsilon(u^{A,k}, y_0^k) - \Theta_\tau^\varepsilon(\bar{u}, \bar{y}_0) &= Dg_i(\tau, \bar{y}_\tau)(z_\tau^{A,k} + z_\tau^2[v^{A,k}, \delta y_0^k]) \\ &\quad + \frac{1}{2} D^2 g_i(\tau, \bar{y}_\tau)(z_\tau^{A,k})^2 - \frac{1}{2} \frac{(D_y g_i^{(1)}(\tau, \bar{y}_\tau) z_\tau^{A,k})^2)}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau)} + o(R_{2,k}^2). \end{aligned} \tag{2.42}$$

Recall that the restriction of μ_i to Δ_τ^ε is a Dirac measure at τ. Summing (2.41) and (2.42), we obtain the following estimate for (2.29b):

$$\begin{aligned} \sum_{\substack{\tau \in \mathcal{T}_{g,i} \\ 1 \le i \le n_g}} \Big[\int_{\Delta_\tau^\varepsilon} & (Dg_i(t, \bar{y}_t) \delta y_t^k + \frac{1}{2} D^2 g_i(t, \bar{y}_t)(z_t^{A,k})^2) \, \mathrm{d}\mu_{i,t} \\ & - \frac{1}{2} \frac{(Dg_i^{(1)}(\tau, \bar{y}_\tau) z_\tau^{A,k})^2)}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau)} \Big] + o(R_{2,k}^2). \end{aligned} \tag{2.43}$$

Combining (2.40) and (2.43), we obtain (2.39). Combining (2.34) and (2.39), we obtain the result. □

2.5 Quadratic growth for bounded strong solutions

We give in this section sufficient second-order optimality conditions in Pontryagin form ensuring the quadratic growth property for bounded strong solutions. Our main result, Theorem 2.5.3, is proved with a classical approach by contradiction.

Assumption 4. There exists $\varepsilon > 0$ such that for all feasible trajectories (u, y) in $(\mathcal{U} \times \mathcal{Y})$ with $\|y - \bar{y}\| \le \varepsilon$, if (u, y) satisfies the mixed constraints, then there exists $\hat{u}$ such that

$$\hat{u}_t \in U(t) \text{ for a.a. } t \quad \text{and} \quad \|u - \hat{u}\|_\infty = O(\|y - \bar{y}\|_\infty).$$

This assumption is a metric regularity property, global in u and local in y. Note that the required property is different from (2.22).

Definition 2.5.1. A quadratic form Q on a Hilbert space X is said to be a *Legendre form* iff it is weakly lower semi-continuous and if it satisfies the following property: if $x^k \rightharpoonup x$ weakly in X and $Q(x^k) \to Q(x)$, then $x^k \to x$ strongly in X.

Assumption 5. For all $\lambda \in \Lambda_P(\bar{u}, \bar{y})$, $\Omega[\lambda]$ is a Legendre form.

Remark 2.5.2. By [20, Lemma 21], this assumption is satisfied if for all $\lambda \in \Lambda_P(\bar{u}, \bar{y})$, there exists $\gamma > 0$ such that for almost all t,

$$\gamma I \leq D^2_{uu} H^a[p^\lambda_t, \nu_t](t, \bar{u}_t, \bar{y}_t) \tag{2.44}$$

where I is the identity matrix. In particular, in the absence of mixed and control constraints, the quadratic growth of the Hamiltonian (2.46) implies (2.44).

For all $R > \|\bar{u}\|_\infty$, we define

$$\begin{aligned}\Lambda^R_P(\bar{u}, \bar{y}) = \ &\{\lambda \in \Lambda_L(\bar{u}, \bar{y}) : \text{for a.a. } t, \text{ for all } u \in U(t) \text{ with } |u| \leq R, \\ &\qquad H[p^\lambda_t](t, u, \bar{y}_t) - H[p^\lambda_t](t, \bar{u}_t, \bar{y}_t) \geq 0\,\}.\end{aligned} \tag{2.45}$$

Note that $\Lambda_P(\bar{u}, \bar{y}) = \cap_{R > \|\bar{u}\|_\infty} \Lambda^R_P(\bar{u}, \bar{y})$. Remember that $C_2(\bar{u}, \bar{y})$ is the critical cone in L^2, defined by (2.12).

Theorem 2.5.3. *Suppose that Assumptions 1-5 hold. If the following second-order sufficient conditions hold: for all $R > \|\bar{u}\|_\infty$,*

1. *there exist $\alpha > 0$ and $\lambda^* \in \Lambda^R_P(\bar{u}, \bar{y})$ such that*

$$\left\{\begin{array}{l}\text{for a.a. } t, \text{ for all } u \in U(t) \text{ with } |u| \leq R, \\ H[p^{\lambda^*}_t](t, u, \bar{y}_t) - H[p^{\lambda^*}_t](t, \bar{u}_t, \bar{y}_t) \geq \alpha |u - \bar{u}_t|^2,\end{array}\right. \tag{2.46}$$

2. *for all $(v, z) \in C_2 \backslash \{0\}$, there exists $\lambda \in \Lambda^R_P(\bar{u}, \bar{y})$ such that $\Omega[\lambda](v, z) > 0$,*

then the quadratic growth property for bounded strong solutions holds at $(\bar{u}, \bar{y})$.

Proof. We prove this theorem by contradiction. Let $R > \|\bar{u}\|_\infty$, let us suppose that there exists a sequence $(u^k, y^k)_k$ of feasible trajectories such that $\|u^k\|_\infty \leq R$, $\|y^k - \bar{y}\|_\infty \to 0$ and

$$\phi(y^k_0, y^k_T) - \phi(\bar{y}_0, \bar{y}_T) \leq o(\|u^k - \bar{u}\|^2_2 + |y^k_0 - \bar{y}_0|^2).$$

We use the notations introduced in (2.25). Let $\lambda^* = (\beta^*, \Psi^*, \nu^*, \mu^*) \in \Lambda^R_P(\bar{u}, \bar{y})$ be such that (2.46) holds.

$\triangleright$ *First step:* $\|u^k - \bar{u}\|_2 = R_{2,k} \to 0$.

By Assumption 4, there exists a sequence of controls $(\hat{u}^k)_k$ such that

$$\hat{u}^k_t \in U(t) \text{ for a.a. } t \quad \text{and} \quad \|u^k - \hat{u}^k\|_\infty = O(\|\delta y^k\|_\infty) = O(R_{1,k}).$$

As a consequence of (2.33), we obtain that

$$\begin{aligned}\beta^*(\phi(y^k_0, y^k_T) - \phi(\bar{y}_0, \bar{y}_T)) \geq &\int_0^T (H[p^{\lambda^*}_t](t, u^k_t, \bar{y}_t) - H[p^{\lambda^*}_t](t, \hat{u}^k_t, \bar{y}_t))\mathrm{d}t \\ &+ \int_0^T (H[p^{\lambda^*}_t](t, \hat{u}^k_t, \bar{y}_t) - H[p^{\lambda^*}_t](t, \bar{u}_t, \bar{y}_t))\mathrm{d}t + o(1) \\ \geq &\ \alpha\|\hat{u}^k - \bar{u}\|^2_2 + o(1) = \alpha\|u^k - \bar{u}\|^2_2 + o(1).\end{aligned}$$

Since $\beta^*(\phi(y^k_0, y^k_T) - \phi(\bar{y}_0, \bar{y}_T)) \to 0$, we obtain that $\|u^k - \bar{u}\|_2 \to 0$. Therefore, the sequence of trajectories satisfy (2.21) and by the Cauchy-Schwarz inequality, $R_{1,k} \to 0$.

Now, we can build a sequence of partitions $(A_k, B_k)_k$ which satisfies (2.24). Let us define

$$A_k := \left\{t \in [0, T], |u^k_t - \bar{u}_t| \leq R^{1/4}_{1,k}\right\}$$

and $B_k := [0,T]\backslash A_k$. Then,

$$\|u^k - \bar{u}\|_1 \geq \int_{B_k} (\|u^k - \bar{u}\|_1 + |\delta y_0^k|)^{1/4} \mathrm{d}t \geq |B_k|(\|u^k - \bar{u}\|_1)^{1/4}.$$

Thus, $|B_k| \leq (\|u^k - \bar{u}\|_1)^{3/4} \to 0$ and we can construct all the elements useful for the decomposition principle: $\tilde{u}^k$, $u^{A,k}$, $v^{A,k}$, δy^k, $y^{A,k}$, and $z^{A,k}$.

Let $\bar{\lambda} \in \Lambda_P^R(\bar{u},\bar{y})$, $\sigma \in [0,1)$ and $\lambda := \sigma\bar{\lambda} + (1-\sigma)\lambda^*$. The Hamiltonian depending linearly on the dual variable, the quadratic growth property (2.46) holds for λ (instead of λ^*) with the coefficient $(1-\sigma)\alpha > 0$ (instead of α).

$\triangleright$ *Second step: we show that $R_{2,B,k} = O(R_{2,A,k})$ and $\Omega[\lambda](v^{A,k}, z^{A,k}) \leq o(R_{2,A,k}^2)$.* By the decomposition principle (Theorem 2.4.2), we obtain that

$$\begin{aligned}\Omega[\lambda](v^{A,k}, z^{A,k}) + \int_{B_k} [H[p_t^\lambda](t, u_t^k, \bar{y}_t) - H[p_t^\lambda](t, \tilde{u}_t^k, \bar{y}_t)] \mathrm{d}t& \\ \leq \beta(\phi(y_0^k, y_T^k) - \phi(\bar{y}_0, \bar{y}_T)) + o(R_{2,k}^2) \leq o(R_{2,k}^2).& \quad (2.47)\end{aligned}$$

We cannot use directly the quadratic growth of the Hamiltonian, since the control u^k does not satisfy necessarily the mixed constraint $c(t, u_t^k, \bar{y}_t) \leq 0$. Therefore, we decompose the difference of Hamiltonians as follows:

$$\Delta_k = \int_{B_k} [H[p_t^\lambda](t, u_t^k, \bar{y}_t) - H[p_t^\lambda](t, \tilde{u}_t^k, \bar{y}_t)] \mathrm{d}t = \Delta_k^a + \Delta_k^b + \Delta_k^c,$$

with

$$\begin{aligned}\Delta_k^a &:= \int_{B_k} [H[p_t^\lambda](t, u_t^k, \bar{y}_t) - H[p_t^\lambda](t, \hat{u}^k, \bar{y}_t)] \mathrm{d}t,\\ \Delta_k^b &:= \int_{B_k} [H[p_t^\lambda](t, \hat{u}^k, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t)] \mathrm{d}t,\\ \Delta_k^c &:= \int_{B_k} [H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) - H[p_t^\lambda](t, \tilde{u}_t, \bar{y}_t)] \mathrm{d}t.\end{aligned}$$

Note first that by (2.47), $\Delta_k \leq O(R_{2,A,k}^2) + o(R_{2,B,k}^2)$. We set

$$\hat{R}_{2,B,k} = \Big[\int_{B_k} |\hat{u}_t^k - \bar{u}_t|^2 \,\mathrm{d}t\Big]^{1/2}.$$

Note that $\Delta_k^b \geq 0$. In order to prove that $R_{2,B,k} = O(R_{2,A,k})$, we need the following two estimates:

$$|\Delta_k^a| + |\Delta_k^c| = o(\Delta_k^b), \quad (2.48)$$

$$|R_{2,B,k}^2 - \hat{R}_{2,B,k}^2| = o(R_{2,B,k}^2). \quad (2.49)$$

Since the control is uniformly bounded, the Hamiltonian is Lipschitz with respect to u and we obtain that

$$|\Delta_k^a| + |\Delta_k^c| = O(|B_k| R_{1,k}),$$

while, as a consequence of the quadratic growth of the Hamiltonian,

$$\begin{aligned}\Delta_k^b &\geq \alpha(1-\mu)\hat{R}_{2,B,k}^2\\ &\geq \alpha(1-\mu)|B_k|\big(R_{1,k}^{1/4} + O(R_{1,k})\big)^2\\ &\geq \alpha(1-\mu)|B_k|R_{1,k}^{1/2}\big(1 + O(R_{1,k}^{3/4})\big)^2, \quad (2.50)\end{aligned}$$

which proves (2.48). Combined with (2.47) and $\Omega[\lambda](v^{A,k}, z^{A,k}) = O(R_{2,A,k}^2)$, we obtain that

$$\Delta_k^b = O(\Delta_k^a + \Delta_k^b + \Delta_k^c) = O(\Delta_k) = O(R_{2,A,k}^2) + o(R_{2,B,k}^2) \tag{2.51}$$

and

$$\hat{R}_{2,B,k}^2 \leq \frac{1}{\alpha(1-\mu)} \Delta_k^b = O(\Delta_k) \leq O(R_{2,A,k}^2) + o(R_{2,B,k}^2). \tag{2.52}$$

Let us prove (2.49). For all k, we have

$$\begin{aligned}
\left| R_{2,B,k}^2 - \hat{R}_{2,B,k}^2 \right| &= \left| \int_{B_k} (|u_t^k - \bar{u}_t|^2 - |\hat{u}_t^k - \bar{u}_t^2|) \mathrm{d}t \right| \\
&\leq \int_{B_k} |u_t^k - \hat{u}_t^k| (|u_t^k - \hat{u}^k| + 2|u_t^k - \bar{u}_t|) \, \mathrm{d}t \\
&\leq \|u^k - \hat{u}^k\|_\infty \Big(\int_{B_k} |u_t^k - \hat{u}_t^k| \, \mathrm{d}t + 2 \int_{B_k} |u_t^k - \bar{u}_t| \, \mathrm{d}t \Big) \\
&= O(R_{1,k})(O(|B_k| R_{1,k}) + O(R_{1,B,k})) \\
&= o(R_{2,k}^2)
\end{aligned}$$

which proves (2.49), by using (2.50). Combined with (2.52), it follows that

$$R_{2,B,k}^2 = \hat{R}_{2,B,k}^2 + o(R_{2,k}^2) = O(R_{2,A,k}^2) + o(R_{2,B,k}^2)$$

and finally that

$$R_{2,B,k}^2 = O(R_{2,A,k}^2) \quad \text{and} \quad R_{2,k} = O(R_{2,A,k}). \tag{2.53}$$

Moreover, since $\Delta_k^b \geq 0$ and by (2.48), (2.51), and (2.53),

$$\Omega[\lambda](v^{A,k}, z^{A,k}) \leq o(R_{2,k}^2) - \Delta_k^a - \Delta_k^c \leq o(R_{2,k}^2) + o(\Delta^k) \leq o(R_{2,A,k}^2). \tag{2.54}$$

▷ *Third step: contradiction.*
Let us set

$$w^k = \frac{v^{A,k}}{R_{2,A,k}} \quad \text{and} \quad x^k = \frac{z^{A,k}}{R_{2,A,k}} = z[w^k, \delta y_0^k / R_{2,A,k}].$$

The sequence $(w^k, x_0^k)_k$ being bounded in $L^2(0,T;\mathbb{R}^m) \times \mathbb{R}^n$, it converges (up to a subsequence) for the weak topology to a limit point, say (w, x_0). Let us set $x = z[w, x_0]$. Let us prove that $(w, x) \in C_2(\bar{u}, \bar{y})$. It follows from the continuity of the linear mapping

$$z : (v, z_0) \in L^2(0,T;\mathbb{R}^m) \times \mathbb{R}^n \mapsto z[v, z_0] \in W^{1,2}(0,T;\mathbb{R}^n)$$

and the compact imbedding of $W^{1,2}(0,T;\mathbb{R}^n)$ into $C(0,T;\mathbb{R}^n)$ that extracting if necessary, $(x^k)_k$ converges uniformly to x. Using (2.26), we obtain that

$$\begin{aligned}
\|\delta y^k - R_{2,A,k} x\|_\infty &= \|z^{A,k} - R_{2,A,k} x\|_\infty + o(R_{2,A,k}) \\
&= R_{2,A,k} (\|x^k - x\|_\infty + o(1)) \\
&= o(R_{2,A,k}).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\phi(y_0^k, y_T^k) - \phi(\bar{y}_0, \bar{y}_T) = R_{2,A,k} D\phi(\bar{y}_0, \bar{y}_T)(x_0, x_T) + o(R_{2,A,k}), \\
&\Phi(y_0^k, y_T^k) - \Phi(\bar{y}_0, \bar{y}_T) = R_{2,A,k} D\phi(\bar{y}_0, \bar{y}_T)(x_0, x_T) + o(R_{2,A,k}), \\
&\left\| g(t, y_t^k) - g(t, \bar{y}_t) - R_{2,A,k} Dg(t, \bar{y}_t) x_t \right\|_\infty = o(R_{2,A,k}).
\end{aligned}$$

This proves that

$$\begin{aligned}&D\phi(\bar{y}_0, \bar{y}_T)(x_0, x_T) = 0,\\&D\Phi(\bar{y}_0, \bar{y}_T)(x_0, x_T) \in T_{K_\Phi}(\phi(\bar{y}_0, \bar{y}_T)),\\&Dg(\cdot, \bar{y})x \in T_{K_g}(g(\cdot, \bar{y})).\end{aligned}$$

Let us set, for a. a. t,

$$\bar{c}_t = c(t, \bar{u}_t, \bar{y}_t) \quad \text{and} \quad c_t^k = \bar{c}_t \mathbf{1}_{\{t \in B_k\}} + c(t, u^{A,k}, y_t^k)\mathbf{1}_{\{t \in A_k\}}.$$

We easily check that

$$\|c_t^k - (\bar{c}_t + R_{2,A,k} Dc(t, \bar{u}_t, \bar{y}_t)(w_t^k, x_t^k))\|_\infty = o(R_{2,A,k}).$$

Therefore,

$$\frac{c^k - \bar{c}}{R_{2,A,k}} \rightharpoonup Dc(t, \bar{u}_t, \bar{y}_t)(w_t, x_t) \tag{2.55}$$

in $L^2(0, T; \mathbb{R}_-^{n_c})$. Moreover, $c_t^k \leq 0$, for almost all t, therefore the ratio in (2.55) belongs to $T_{K_c}(c(\cdot, \bar{u}, \bar{y}))$. This cone being closed and convex, it is weakly closed and we obtain finally that

$$Dc(t, \bar{u}_t, \bar{y}_t)(w_t, x_t) \in T_{K_c}(c(\cdot, \bar{u}, \bar{y})).$$

We have proved that $(w, x) \in C_2(\bar{u}, \bar{y})$. By Assumption 5, $\Omega[\lambda]$ is weakly$*$ lower semi-continuous, thus from (2.54) we get

$$\Omega[\lambda](w, x) \leq \liminf_k \Omega[\lambda](w^k, x^k) \leq 0.$$

Passing to the limit when $\mu \to 1$, we find that $\Omega[\bar{\lambda}](w, x) \leq 0$. Since $\bar{\lambda}$ was arbitrary in $\Lambda_P^R(\bar{u}, \bar{y})$, it follows by the sufficient conditions that $(w, x) = 0$ and that for any λ for which the quadratic growth of the Hamiltonian holds,

$$\Omega[\lambda](w, x) = \lim_k \Omega[\lambda](w^k, x^k).$$

Since $\Omega[\lambda]$ is a Legendre form, we obtain that $(w^k, x^k)_k$ converges strongly to 0, in contradiction with the fact that $\|w^k\|_2 + |x_0^k| = 1$. This concludes the proof of the theorem. □

2.6 Characterization of quadratic growth

In this section, we prove that the second-order sufficient conditions are also necessary to ensure the quadratic growth property. The proof relies on the necessary second-order optimality conditions in Pontryagin form that we established in [18]. Let us first remember the assumptions required to use these necessary conditions.

Assumption 6. The mappings f and g are C^∞, c is uniformly quasi-C^2, and Φ^E, Φ^I and ϕ are C^2.

For $\delta \geq 0$ and $\varepsilon > 0$, let us define

$$\begin{aligned}\Delta_{c,i}^\delta &:= \{t \in [0, T] \,:\, c_i(t, \bar{u}_t, \bar{y}_t) \geq -\delta\},\\\Delta_{g,i}^0 &:= \{t \in [0, T] \,:\, g_i(t, \bar{y}_t) = 0\} \setminus \mathcal{T}_{g,i},\\\Delta_{g,i}^\varepsilon &:= \{t \in [0, T] \,:\, \operatorname{dist}(t, \Delta_{g,i}^0) \leq \varepsilon\}.\end{aligned}$$

Assumption 7 is a geometrical assumption on the structure of the control. Assumption 8 is related to the controllability of the system, see [22, Lemma 2.3] for conditions ensuring this property.

Assumption 7. For $1 \leq i \leq n_g$, $\Delta^0_{g,i}$ has finitely many connected components and g_i is of finite order q_i.

Assumption 8. There exist $\delta', \varepsilon' > 0$ such that the linear mapping from $\mathcal{V}_2 \times \mathbb{R}^n$ to $\prod_{i=1}^{n_c} L^2(\Delta^{\delta'}_{c,i}) \times \prod_{i=1}^{n_g} W^{q_i,2}(\Delta^{\varepsilon'}_{g,i})$ defined by

$$(v, z^0) \mapsto \begin{pmatrix} \left(Dc_i(\cdot, \bar{u}, \bar{y})(v, z[v, z^0])|_{\Delta^{\delta'}_{c,i}} \right)_{1 \leq i \leq n_c} \\ \left(Dg_i(\cdot, \bar{y}) z[v, z^0]|_{\Delta^{\varepsilon'}_{g,i}} \right)_{1 \leq i \leq n_g} \end{pmatrix} \quad \text{is onto.}$$

The second-order necessary conditions are satisfied on a subset of the critical cone called *strict critical cone*. The following assumption ensures that the two cones are equal [25, Proposition 3.10].

Assumption 9. There exists $\lambda = (\bar{\beta}, \bar{\Psi}, \bar{\nu}, \bar{\mu}) \in \Lambda_L(\bar{u}, \bar{y})$ such that

$$\begin{aligned} &\bar{\nu}_i(t) > 0 \quad \text{for a.a. } t \in \Delta^0_{c,i} && 1 \leq i \leq n_c, \\ &\operatorname{supp}(\bar{\mu}_i) \cap \Delta^0_{g,i} = \Delta^0_{g,i} && 1 \leq i \leq n_g. \end{aligned}$$

The main result of [18] was the following necessary conditions in Pontryagin form:

Theorem 2.6.1. *Let Assumptions 2, 3, and 6-9 hold. If $(\bar{u}, \bar{y})$ is a Pontryagin minimum of problem (P), then for any $(v, z) \in C_2(\bar{u}, \bar{y})$, there exists $\lambda \in \Lambda_P(\bar{u}, \bar{y})$ such that*

$$\Omega[\lambda](v, z) \geq 0.$$

Assumption 10. All Pontryagin multipliers $\lambda = (\beta, \Psi, \nu, \mu)$ are non singular, that is to say, are such that $\beta > 0$.

This assumption is satisfied if one of the usual qualification conditions holds since then, all Lagrange multipliers are non singular. In [18, Proposition A.13], we gave a weaker condition ensuring the non singularity of Pontryagin multipliers.

Lemma 2.6.2. *Let Assumptions 2, 3, and 6-10 hold. If the quadratic growth property for bounded strong solutions holds at $(\bar{u}, \bar{y})$, then the sufficient second-order conditions are satisfied.*

Proof. Let $R > \|\bar{u}\|_\infty$, let $\alpha > 0$ and $\varepsilon > 0$ be such that for all $(u, y) \in F(P)$ with $\|u\|_\infty \leq R$ and $\|y - \bar{y}\|_\infty \leq \varepsilon$,

$$\phi(y_0, y_T) - \phi(\bar{y}_0, \bar{y}_T) \geq \alpha(\|u - \bar{u}\|_2^2 + |y_0 - \bar{y}_0|^2).$$

Then, $(\bar{u}, \bar{y})$ is a Pontryagin minimum to a new optimal control problem with cost

$$\phi(y_0, y_T) - \alpha(|y_0 - \bar{y}_0|^2 + \|u - \bar{u}\|^2)$$

and with the additional constraint $\|u\|_\infty \leq R$. The new Hamiltonian and the new Hessian of the Lagrangian are now given by resp.

$$H[p](t, u, y) - \alpha\beta|u - \bar{u}|^2 \quad \text{and} \quad \Omega[\lambda](v, z) - \alpha\beta(\|v\|^2 + |z_0|^2).$$

It is easy to check that the costate equation is unchanged and that the set of Lagrange multipliers of both problems are the same. The set of Pontryagin multipliers to the new problem is the set of Lagrange multipliers λ for which for a.a. t, for all $u \in U(t)$ with $|u| \leq R$,

$$H[p_t^\lambda](t, u, \bar{y}_t) - H[p_t^\lambda](t, \bar{u}_t, \bar{y}_t) \geq \alpha\beta|u - \bar{u}|_2^2,$$

it is thus included into $\Lambda_P^R(\bar{u}, \bar{y})$ (which was defined by (2.45)). Let (v, z) in $C_2(\bar{u}, \bar{y})\backslash\{0\}$, then by Theorem 2.6.1, there exists a Pontryagin multiplier (to the new problem), belonging to $\Lambda_P^R(\bar{u}, \bar{y})$, which is such that

$$\Omega[\lambda](v, z) \geq \alpha\beta(|z_0|^2 + \|v\|_2^2) > 0.$$

The sufficient second-order optimality conditions are satisfied. □

Finally, combining Theorem 2.5.3 and Lemma 2.6.2 we obtain a characterization of the quadratic growth property for bounded strong solutions (under the Legendre-Clebsch assumption).

Theorem 2.6.3. *Let Assumptions 2-10 hold. Then, the quadratic growth property for bounded strong solutions holds at $(\bar{u}, \bar{y})$ if and only if the sufficient second-order conditions are satisfied.*

Part II

Optimal control of differential equations with memory

Chapter 3

Optimality conditions for integral equations

This chapter is taken from [17]:

J.F. Bonnans, C. de la Vega, X. Dupuis. *First and second order optimality conditions for optimal control problems of state constrained integral equations.* J. Optim. Theory Appl. 159(1):1-40, 2013.

This paper deals with optimal control problems of integral equations, with initial-final and running state constraints. The order of a running state constraint is defined in the setting of integral dynamics, and we work here with constraints of arbitrary high orders. First and second-order necessary conditions of optimality are obtained, as well as second-order sufficient conditions. Second-order necessary conditions are expressed by the nonnegativity of the supremum of some quadratic forms. Second-order sufficient conditions are also obtained, in the case where these quadratic forms are of Legendre type.

3.1 Introduction

The dynamics in the optimal control problems we consider in this paper is given by an integral equation. Such equations, sometimes called nonlinear Volterra integral equations, belong to the family of equations with memory and thus are found in many models. Among the fields of application of these equations are population dynamics in biology and growth theory in economy: see [90] or its translation in [84] for one of the first use of integral equations in ecology in 1927 by Volterra, who contributed earlier to their theoretical study [89]; in 1976, Kamien and Muller model the capital replacement problem by an optimal control problem with an integral state equation [57]. First-order optimality conditions for such problems were known under the form of a maximum principle since Vinokurov's paper [88] in 1967, translated in 1969 [88] and whose proof has been questionned by Neustadt and Warga [70] in 1970. Maximum principles have then been provided by Bakke [8], Carlson [28], or more recently de la Vega [35] for an optimal terminal time control problem. First-order optimality conditions for control problems of the more general family of equations with memory are obtained by Carlier and Tahraoui [27].

None of the previously cited articles consider what we will call 'running state constraints'. That is what Bonnans and de la Vega did in [16], where they provide Pontryagin's principle, i.e. first-order optimality conditions. In this work we are particularly interested in second-order necessary conditions, in presence of running state constraints. Such constraints drive to optimization problems with inequality constraints in the infinite-dimensional space of continuous functions. Thus second-order necessary conditions on a so-called *critical cone* will contain an extra term, as it has been discovered in 1988 by Kawasaki [58] and generalized in 1990 by Cominetti [34], in an abstract setting. It is possible to compute this extra term in the case of state constrained optimal control problems; this is what is done by Páles and Zeidan [75] or Bonnans and Hermant [20, 22] in the framework of ODEs.

Our strategy here is different and follows [21], with the differences that we work with integral equations and that we add initial-final state constraints which lead to nonunique Lagrange multipliers. The idea was already present in [58] and is closely related to the concept of extended polyhedricity [25]: the extra term mentioned above vanishes if we write second-order necessary conditions on a subset of the critical cone, the so-called *radial critical cone*. This motivates to introduce an auxiliary optimization problem, the *reduced problem*, for which under some assumptions the radial critical cone is dense in the critical cone. Optimality conditions for the reduced problem are relevant for the original problem and the extra term now appears as the derivative of a new constraint in the reduced problem. We will devote a lot of effort to the proof of the density result and we will mention a flaw in [21] concerning this proof.

The paper is organized as follows. We set the optimal control problem, define Lagrange multipliers and work on the notion of order of a running state constraint in our setting in Section 3.2. The reduced problem is introduced in Section 3.3, followed by first-order necessary conditions and second-order necessary conditions on the radial critical cone. The main results are presented in Section 3.4. After some specific assumptions, we state and prove the technical Lemma 3.4.2 which is then used to strengthen the first-order necessary conditions already obtained and to get the density result that we need. With this density result, we obtain second-order necessary conditions on the critical cone. Second-order suficient conditions are also given in this section. Some of the technical aspects are postponed in the appendix.

Notations We denote by h_t the value of a function h at time t if h depends only on t, and by $h_{i,t}$ its ith component if h is vector-valued. To avoid confusion we denote partial derivatives of a function h of (t,x) by $D_t h$ and $D_x h$, and we keep the symbol D without any subscript for the differentiation w.r.t. all variables. We identify the dual space of $\mathbb{R}^n$ with the space $\mathbb{R}^{n*}$ of n-dimensional horizontal vectors. Generally, we denote by X^* the dual space of a topological vector space X. We use $|\cdot|$ for both the Euclidean norm on finite-dimensional vector spaces and for the cardinal of finite sets, $\|\cdot\|_s$ and $\|\cdot\|_{q,s}$ for the standard norms on the Lesbesgue spaces L^s and the Sobolev spaces $W^{q,s}$, respectively.

3.2 Optimal control of state constrained integral equations

3.2.1 Setting

We consider an optimal control problem with running and initial-final state constraints, of the following type:

$$(P) \qquad \min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \int_0^T \ell(u_t,y_t)\mathrm{d}t + \phi(y_0,y_T) \tag{3.1}$$

$$\text{subject to} \qquad y_t = y_0 + \int_0^t f(t,s,u_s,y_s)\mathrm{d}s, \quad t\in[0,T], \tag{3.2}$$

$$g(y_t) \le 0, \quad t\in[0,T], \tag{3.3}$$

$$\Phi^E(y_0,y_T) = 0, \tag{3.4}$$

$$\Phi^I(y_0,y_T) \le 0, \tag{3.5}$$

where

$$\mathcal{U} := L^\infty([0,T];\mathbb{R}^m), \quad \mathcal{Y} := W^{1,\infty}([0,T];\mathbb{R}^n)$$

are the *control space* and the *state space*, respectively.

The data are $\ell\colon \mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}$, $\phi\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $f\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^n$, $g\colon\mathbb{R}^n\to\mathbb{R}^r$, $\Phi^E\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{s_E}$, $\Phi^I\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{s_I}$ and $T>0$. We make the following assumption:

(A0) $\ell,\phi,f,g,\Phi^E,\Phi^I$ are of class C^∞ and f is Lipschitz.

Remark 3.2.1.
1. We set τ as the symbol for the first variable of f. Observe that if $D_\tau f\equiv 0$, we recover an optimal control problem of a state constrained ODE. More generally, if $D^d_{\tau^d} f\equiv 0$, then the integral equation (3.2) can be written as a system of controlled differential equations by adding $d-1$ state variables.

2. The running cost ℓ and the running state constraints g appear in some applications as functions of (t,u,y) and (t,y), respectively. It fits our framework if ℓ and g are of class C^∞ w.r.t. all variables by adding a state variable, but the case where they are not regular w.r.t. t is not treated here.

We call *trajectory* a pair $(u,y)\in\mathcal{U}\times\mathcal{Y}$ which satisfies the *state equation* (3.2). Under assumption (A0) it can be shown by standard contraction arguments that for any $(u,y_0)\in\mathcal{U}\times\mathbb{R}^n$, the state equation (3.2) has a unique solution y in $\mathcal{Y}$, denoted by $y[u,y_0]$. Moreover, the map $\Gamma\colon\mathcal{U}\times\mathbb{R}^n\to\mathcal{Y}$ defined by $\Gamma(u,y_0):=y[u,y_0]$ is of class C^∞.

3.2.2 Lagrange multipliers

The dual space of the space of vector-valued continuous functions $C([0,T];\mathbb{R}^r)$ is the space of finite vector-valued Radon measures $\mathcal{M}([0,T];\mathbb{R}^{r*})$, under the pairing

$$\langle \mu, h\rangle := \int_{[0,T]} \mathrm{d}\mu_t h_t = \sum_{1\leq i\leq r} \int_{[0,T]} h_{i,t}\mathrm{d}\mu_{i,t.}$$

We define $BV([0,T];\mathbb{R}^{n*})$, the space of vector-valued functions of bounded variations, as follows: let I be an open set which contains $[0,T]$; then

$$BV([0,T];\mathbb{R}^{n*}) := \left\{h \in L^1(I;\mathbb{R}^{n*}) \,:\, Dh \in \mathcal{M}(I;\mathbb{R}^{n*}),\, \mathrm{supp}(Dh) \subset [0,T]\right\},$$

where Dh is the distributional derivative of h; if h is of bounded variations, we denote it by $\mathrm{d}h$. For $h \in BV([0,T];\mathbb{R}^{n*})$, there exists $h_0, h_T \in \mathbb{R}^{n*}$ such that

$$\begin{aligned} h &= h_0 \quad \text{a.e. on }]-\infty,0[\cap I, \\ h &= h_T \quad \text{a.e. on }]T,+\infty[\cap I. \end{aligned} \tag{3.6}$$

Conversely, we can identify any measure $\mu \in \mathcal{M}([0,T];\mathbb{R}^{r*})$ with the derivative of a function of bounded variations, denoted again by μ, such that $\mu_T = 0$. See Appendix 3.A.1 for more details.

Let

$$\mathcal{M} := \mathcal{M}([0,T];\mathbb{R}^{r*}), \quad \mathcal{P} := BV([0,T];\mathbb{R}^{n*}).$$

We define for $p \in \mathcal{P}$ the *Hamiltonian* $H[p]\colon \mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^n \to \mathbb{R}$ by

$$H[p](t,u,y) := \ell(u,y) + p_t f(t,t,u,y) + \int_t^T p_s D_\tau f(s,t,u,y)\mathrm{d}s \tag{3.7}$$

and for $\Psi \in \mathbb{R}^{s*}$ the *end points Lagrangian* $\Phi[\Psi]\colon \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ by

$$\Phi[\Psi](y_1,y_2) := \phi(y_1,y_2) + \Psi\Phi(y_1,y_2) \tag{3.8}$$

where $s := s_E + s_I$ and $\Phi := (\Phi^E, \Phi^I)$. We also denote $K := \{0\}_{s_E} \times \mathbb{R}^{s_I}_-$, so that (3.4)-(3.5) can be rewritten as $\Phi(y_0,y_T) \in K$. The *normal cone* to K at a point $\bar{\Phi}$, denoted by $N_K(\bar{\Phi})$ and defined as the polar cone of the tangent cone $T_K(\bar{\Phi})$, has here the following characterization: $\Psi \in N_K(\bar{\Phi})$ iff

$$\bar{\Phi} \in K, \quad \Psi_i \geq 0, \quad \Psi_i\bar{\Phi}_i = 0, \quad i = s_E+1,\ldots,s_E+s_I. \tag{3.9}$$

Given a trajectory (u,y) and $(\mu,\Psi) \in \mathcal{M}\times\mathbb{R}^{s*}$, the *adjoint state* p, whenever it exists, is defined as the solution in $\mathcal{P}$ of

$$\begin{cases} -\mathrm{d}p_t = D_y H[p](t,u_t,y_t)\mathrm{d}t + \mathrm{d}\mu_t g'(y_t), \\ (-p_0, p_T) = D\Phi[\Psi](y_0,y_T). \end{cases} \tag{3.10}$$

Note that $\mathrm{d}\mu_t g'(y_t) = \sum_{i=1}^r \mathrm{d}\mu_{i,t} g_i'(y_t)$. The adjoint state does not exist in general, but when it does it is unique. More precisely, we have:

Lemma 3.2.2. *There exists a unique solution in $\mathcal{P}$ of the adjoint state equation with final condition only (i.e. without initial condition):*

$$\begin{cases} -\mathrm{d}p_t = D_y H[p](t,u_t,y_t)\mathrm{d}t + \mathrm{d}\mu_t g'(y_t), \\ p_T = D_{y_2}\Phi[\Psi](y_0,y_T). \end{cases} \tag{3.11}$$

Proof. The contraction argument is given in Appendix 3.A.1. □

We can now define Lagrange multipliers for optimal control problems in our setting:

Definition 3.2.3. The triple $(\mu, \Psi, p) \in \mathcal{M} \times \mathbb{R}^{s*} \times \mathcal{P}$ is a *Lagrange multiplier* associated with $(\bar{u}, \bar{y})$ iff

$$p \text{ is the adjoint state associated with } (\bar{u}, \bar{y}, \mu, \Psi), \tag{3.12}$$

$$\mu \geq 0, \quad g(\bar{y}) \leq 0, \quad \int_{[0,T]} \mathrm{d}\mu_t g(\bar{y}_t) = 0, \tag{3.13}$$

$$\Psi \in N_K \left(\Phi(\bar{y}_0, \bar{y}_T) \right), \tag{3.14}$$

$$D_u H[p](t, \bar{u}_t, \bar{y}_t) = 0 \text{ for a.a. } t \in [0, T]. \tag{3.15}$$

3.2.3 Linearized state equation

For $s \in [1, \infty]$, let

$$\mathcal{V}_s := L^s([0,T]; \mathbb{R}^m), \quad \mathcal{Z}_s := W^{1,s}([0,T]; \mathbb{R}^n).$$

Given a trajectory (u, y) and $(v, z_0) \in \mathcal{V}_s \times \mathbb{R}^n$, we consider the *linearized state equation* in $\mathcal{Z}_s$:

$$z_t = z_0 + \int_0^t D_{(u,y)} f(t, s, u_s, y_s)(v_s, z_s) \mathrm{d}s. \tag{3.16}$$

It is easily shown that there exists a unique solution $z \in \mathcal{Z}_s$ of (3.16), called the *linearized state* associated with the trajectory (u, y) and the direction (v, z_0), and denoted by $z[v, z_0]$ (keeping in mind the nominal trajectory).

Lemma 3.2.4. *There exists $C > 0$ and $C_s > 0$ for any $s \in [1, \infty]$ (depending on (u, y)) such that, for all $(v, z_0) \in \mathcal{V}_s \times \mathbb{R}^n$ and all $t \in [0, T]$,*

$$|z[v, z_0]_t| \leq C \left(|z_0| + \int_0^t |v_s| \mathrm{d}s \right), \tag{3.17}$$

$$\|z[v, z_0]\|_{1,s} \leq C_s \left(|z_0| + \|v\|_s \right). \tag{3.18}$$

Proof. (3.17) comes from Gronwall's lemma and (3.18) from (3.17). □

For $s = \infty$, the linearized state equation arises naturally: let $(u, y_0) \in \mathcal{U} \times \mathbb{R}^n$, $y := \Gamma(u, y_0) \in \mathcal{Y}$. We consider the linearized state associated with the trajectory (u, y) and a direction $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$. Then

$$z[v, z_0] = D\Gamma(u, y_0)(v, z_0).$$

Similarly we can define the *second-order linearized state* $z^2[v, z_0]$ as the unique solution in $\mathcal{Z}_{s/2}$ of

$$z_t^2 = \int_0^t \left(D_y f(t, s, u_s, y_s) z_s^2 + D^2_{(u,y)^2} f(t, s, u_s, y_s)(v_s, z[v, z_0]_s)^2 \right) \mathrm{d}s \tag{3.19}$$

for $(v, z_0) \in \mathcal{V}_s \times \mathbb{R}^n$ and $s \in [2, \infty]$. If $s = \infty$, then

$$z^2[v, z_0] = D^2\Gamma(u, y_0)(v, z_0)^2.$$

3.2.4 Running state constraints

The running state constraints g_i, $i = 1, \ldots, r$, are considered along trajectories (u, y). They produce functions of one variable, $t \mapsto g_i(y_t)$, which belong to $W^{1,\infty}([0, T])$ *a priori* and satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} g_i(y_t) = g_i'(y_t) \left(f(t, t, u_t, y_t) + \int_0^t D_\tau f(t, s, u_s, y_s) \mathrm{d}s \right). \tag{3.20}$$

There are two parts in this derivative:

– $t \mapsto g_i'(y_t) f(t, t, u_t, y_t)$, where u appears pointwisely.

– $t \mapsto g_i'(y_t) \int_0^t D_\tau f(t, s, u_s, y_s) \mathrm{d}s$, where u appears in an integral.

Below we will distinguish these two behaviors and set $\tilde{u}$ as the symbol for the pointwise variable, u for the integral variable (similarly for y). If there is no dependance on $\tilde{u}$, one can differentiate again (3.20) w.r.t. t. This motivates the definition of a notion of total derivative that always "forget" the dependence on $\tilde{u}$. Let us do that formally.

First we need a set which is stable by operations such as in (3.20), so that it will contain the derivatives of any order. It is also of interest to know how the functions we consider depend on $(u, y) \in \mathcal{U} \times \mathcal{Y}$. To answer this double issue, we define the following commutative ring:

$$\mathcal{S} := \left\{ h \,:\, h(t, \tilde{u}, \tilde{y}, u, y) = \sum_\alpha a_\alpha(t, \tilde{u}, \tilde{y}) \prod_\beta \int_0^t b_{\alpha,\beta}(t, s, u_s, y_s) \mathrm{d}s \right\}, \tag{3.21}$$

where $(t, \tilde{u}, \tilde{y}, u, y) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{Y}$, the a_α, $b_{\alpha,\beta}$ are real functions of class C^∞, the sum and the products are finite and an empty product is equal to 1. The following is straightforward:

Lemma 3.2.5. *Let $h \in \mathcal{S}$, $(u, y) \in \mathcal{U} \times \mathcal{Y}$. There exists $C > 0$ such that, for a.a. $t \in [0, T]$ and for all $(\tilde{v}, \tilde{z}, v, z) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{Y}$,*

$$\left| D_{(\tilde{u},\tilde{y},u,y)} h(t, u_t, y_t, u, y)(\tilde{v}, \tilde{z}, v, z) \right| \leq C \left(|\tilde{v}| + |\tilde{z}| + \int_0^t (|v_s| + |z_s|) \, \mathrm{d}s \right).$$

Next we define the derivation $D^{(1)} \colon \mathcal{S} \longrightarrow \mathcal{S}$ as follows (recall that we set τ as the symbol for the first variable of f or b):

1. for $h \colon (t, \tilde{u}, \tilde{y}) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \mapsto a(t, \tilde{u}, \tilde{y}) \in \mathbb{R}$,

$$\begin{aligned}\left(D^{(1)} h \right)(t, \tilde{u}, \tilde{y}, u, y) &:= D_t a(t, \tilde{u}, \tilde{y}) \\ &\quad + D_{\tilde{y}} a(t, \tilde{u}, \tilde{y}) \left(f(t, t, \tilde{u}, \tilde{y}) + \int_0^t D_\tau f(t, s, u_s, y_s) \mathrm{d}s \right).\end{aligned}$$

2. for $h \colon (t, u, y) \in \mathbb{R} \times \mathcal{U} \times \mathcal{Y} \mapsto \int_0^t b(t, s, u_s, y_s) \mathrm{d}s \in \mathbb{R}$,

$$\left(D^{(1)} h \right)(t, \tilde{u}, \tilde{y}, u, y) := b(t, t, \tilde{u}, \tilde{y}) + \int_0^t D_\tau b(t, s, u_s, y_s) \mathrm{d}s.$$

3. for any $h_1, h_2 \in \mathcal{S}$,

$$\begin{aligned}\left(D^{(1)}(h_1 + h_2) \right) &= \left(D^{(1)} h_1 \right) + \left(D^{(1)} h_2 \right), \\ \left(D^{(1)}(h_1 h_2) \right) &= \left(D^{(1)} h_1 \right) h_2 + h_1 \left(D^{(1)} h_2 \right).\end{aligned}$$

It is clear that $D^{(1)}h \in \mathcal{S}$ for any $h \in \mathcal{S}$. The following formula, which is easily checked on $h = a(t, \tilde{u}, \tilde{y})$ and $h = \int_0^t b(t, s, u_s, y_s)\mathrm{d}s$, will be used for any $h \in \mathcal{S}$:

$$\left(D^{(1)}h\right)(t, u_t, y_t, u, y) = D_t h(t, u_t, y_t, u, y) + D_{\tilde{y}} h(t, u_t, y_t, u, y) f(t, t, u_t, y_t) \\ + D_{\tilde{y}} h(t, u_t, y_t, u, y) \int_0^t D_\tau f(t, s, u_s, y_s)\mathrm{d}s. \quad (3.22)$$

Let us now highlight two important properties of $D^{(1)}$. First, it is a notion of total derivative:

Lemma 3.2.6. *Let $h \in \mathcal{S}$ be such that $D_{\tilde{u}} h \equiv 0$, $(u, y) \in \mathcal{U} \times \mathcal{Y}$ be a trajectory and*

$$\varphi \colon t \mapsto h(t, u_t, y_t, u, y).$$

Then $\varphi \in W^{1,\infty}([0, T])$ and

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) = \left(D^{(1)}h\right)(t, u_t, y_t, u, y). \quad (3.23)$$

Proof. We write h as in (3.21). If $D_{\tilde{u}} h \equiv 0$, then for any $u_0 \in \mathbb{R}^m$,

$$\varphi(t) = h(t, u_0, y_t, u, y) \quad (3.24)$$

$$= \sum_\alpha a_\alpha(t, u_0, y_t) \prod_\beta \int_0^t b_{\alpha,\beta}(t, s, u_s, y_s)\mathrm{d}s. \quad (3.25)$$

By (3.25), $\varphi \in W^{1,\infty}([0, T])$. And by (3.24),

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) = D_t h(t, u_0, y_t, u, y) + D_{\tilde{y}} h(t, u_0, y_t, u, y)\dot{y}_t \\ = D_t h(t, u_t, y_t, u, y) + D_{\tilde{y}} h(t, u_t, y_t, u, y)\dot{y}_t$$

since $D_{\tilde{u}} D_t h \equiv D_t D_{\tilde{u}} h \equiv 0$ and $D_{\tilde{u}} D_{\tilde{y}} h \equiv 0$. Using the expression of $\dot{y}_t$ and (3.22), we recognize (3.23). $\square$

Second, it satisfies a principle of commutation with the linearization:

Lemma 3.2.7. *Let h and (u, y) be as in Lemma 3.2.6. Let $s \in [1, \infty]$, $(v, z_0) \in \mathcal{V}_s \times \mathbb{R}^n$, $z := z[v, z_0] \in \mathcal{Z}_s$ and*

$$\psi \colon t \mapsto D_{(\tilde{y},u,y)} h(t, u_t, y_t, u, y)(z_t, v, z).$$

Then $\psi \in W^{1,s}([0, T])$ and

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t) = D_{(\tilde{u},\tilde{y},u,y)} \left[\left(D^{(1)}h\right)(t, u_t, y_t, u, y)\right](v_t, z_t, v, z). \quad (3.26)$$

Proof. Using $D_{\tilde{u}} D_{(\tilde{y},u,y)} h \equiv 0$, we have

$$\psi(t) = D_{(\tilde{y},u,y)} h(t, u_0, y_t, u, y)(z_t, v, z) \\ = \sum_\alpha D_{\tilde{y}} a_\alpha(t, u_0, y_t) z_t \prod_\beta \int_0^t b_{\alpha,\beta}\mathrm{d}s \\ + \sum_{\alpha,\beta} a_\alpha(t, u_0, y_t) \int_0^t D_{(u,y)} b_{\alpha,\beta}(t, s, u_s, y_s)(v_s, z_s)\mathrm{d}s \prod_{\beta' \neq \beta} \int_0^t b_{\alpha,\beta'}\mathrm{d}s.$$

It implies that $\psi \in W^{1,s}([0,T])$ and that

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t) = D^2_{t,(\tilde{y},u,y)}h(t,u_t,y_t,u,y)(z_t,v,z) \\ + D^2_{\tilde{y},(\tilde{y},u,y)}h(t,u_t,y_t,u,y)\left(\dot{y}_t,(z_t,v,z)\right) + D_{\tilde{y}}h(t,u_t,y_t,u,y)\dot{z}_t.$$

On the other hand, we differentiate $D^{(1)}h$ w.r.t. $(\tilde{u},\tilde{y},u,y)$ using (3.22). Then with the expressions of $\dot{y}_t$ and $\dot{z}_t$, we get the relation (3.26). □

The same principle is true at the second-order:

Lemma 3.2.8. *Let h and (u,y) be as in Lemma 3.2.6. Let $s \in [2,\infty]$, $(v,z_0) \in \mathcal{V}_s \times \mathbb{R}^n$, $z := z[v,z_0], \in \mathcal{Z}_s$, $z^2 := z^2[v,z_0] \in \mathcal{Z}_{s/2}$ and*

$$\phi\colon t \mapsto D^2_{(\tilde{y},u,y)^2}h(t,u_t,y_t,u,y)(z_t,v,z)^2 + D_{(\tilde{y},y)}h(t,u_t,y_t,u,y)(z^2_t,z^2).$$

Then $\phi \in W^{1,s/2}([0,T])$ and

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}(t) = D^2_{(\tilde{u},\tilde{y},u,y)^2}\left[\left(D^{(1)}h\right)(t,u_t,y_t,u,y)\right](v_t,z_t,v,z)^2 \\ + D_{(\tilde{y},y)}\left[\left(D^{(1)}h\right)(t,u_t,y_t,u,y)\right](z^2_t,z^2).$$

Proof. We apply the definitions and the results of this section to a problem where the control variables are (u,v), the state variables are (y,z) and the dynamics is given by

$$\begin{cases} y_t = y_0 + \displaystyle\int_0^t f(t,s,u_s,y_s)\mathrm{d}s, \\ z_t = z_0 + \displaystyle\int_0^t D_{(u,y)}f(t,s,u_s,y_s)(v_s,z_s)\mathrm{d}s. \end{cases}$$

Note that the linearized dynamics at (u,v,y,z) in the direction $(v,0,z_0,0)$ is given by

$$\begin{cases} z_t = z_0 + \displaystyle\int_0^t D_{u,y)}f(t,s,u_s,y_s)(v_s,z_s)\mathrm{d}s, \\ z^2_t = \displaystyle\int_0^t \left(D_y f(t,s,u_s,y_s)z^2_s + D^2(u,y)^2 f(t,s,u_s,y_s)(v_s,z_s)^2\right)\mathrm{d}s. \end{cases}$$

Let H be defined by

$$H(t,\tilde{u},\tilde{v},\tilde{y},\tilde{z},u,v,y,z) := D_{(\tilde{u},\tilde{y},u,y)}h(t,\tilde{u},\tilde{y},u,y)(\tilde{v},\tilde{z},v,z).$$

If $D_{\tilde{u}}h \equiv 0$, then $D_{(\tilde{u},\tilde{v})}H \equiv 0$ and

$$D_{(\tilde{y},\tilde{z},u,v,y,z)}H(t,u_t,v_t,y_t,z_t,u,v,y,z)(z_t,z^2_t,v,0,z,z^2) \\ = D^2_{(\tilde{y},u,y)^2}h(t,u_t,y_t,u,y)(z_t,v,z)^2 + D_{(\tilde{y},y)}h(t,u_t,y_t,u,y)(z^2_t,z^2).$$

By Lemma 3.2.7, the time derivative of this function is

$$D_{(\tilde{u},\tilde{v},\tilde{y},\tilde{z},u,v,y,z)}\left[(D^{(1)}H)(t,u_t,v_t,y_t,z_t,u,v,y,z)\right](v_t,0,z_t,z^2_t,v,0,z,z^2), \tag{3.27}$$

and by Lemma 3.2.6, definition of H and Lemma 3.2.7 again, we get successively

$$\begin{aligned}(D^{(1)}H)(t,u_t,v_t,y_t,z_t,u,v,y,z) &= \frac{\mathrm{d}}{\mathrm{d}t}H(t,u_t,v_t,y_t,z_t,u,v,y,z),\\ &= \frac{\mathrm{d}}{\mathrm{d}t}D_{(\tilde{y},u,y)}h(t,u_t,y_t,u,y)(z_t,v,z),\\ &\qquad = D_{(\tilde{u},\tilde{y},u,y)}\left[(D^{(1)}h)(t,u_t,y_t,u,y)\right](v_t,z_t,v,z).\end{aligned}$$

Then equation (3.27) is equal to

$$\begin{aligned}D^2_{(\tilde{u},\tilde{y},u,y)^2}\left[\left(D^{(1)}h\right)(t,u_t,y_t,u,y)\right](v_t,z_t,v,z)^2\\ + D_{(\tilde{y},y)}\left[\left(D^{(1)}h\right)(t,u_t,y_t,u,y)\right](z_t^2,z^2)\end{aligned}$$

and Lemma 3.2.8 is proved. □

Finally we define the order of a running state constraint g_i. Let $g_i^{(0)} := g_i$ and $g_i^{(j+1)} := D^{(1)}g_i^{(j)}$. Note that $g_i \in \mathcal{S}$, so $g_i^{(j)} \in \mathcal{S}$ for all $j \geq 0$. Moreover, if we write $g_i^{(j)}$ as in (3.21), the a_α and $b_{\alpha,\beta}$ are combinations of derivatives of f and g_i.

Definition 3.2.9. The *order* of the constraint g_i is the greatest positive integer q_i such that

$$D_{\tilde{u}}g_i^{(j)} \equiv 0 \quad \text{for all } j = 0,\ldots,q_i - 1.$$

We have a result similar to Lemma 9 in [20], but now for integral dynamics and up to the second-order. Let $(u,y) \in \mathcal{U}\times\mathcal{Y}$ be a trajectory, $(v,z_0) \in \mathcal{V}_s \times \mathbb{R}^n$, $z := z[v,z_0] \in \mathcal{Z}_s$ and $z^2 := z^2[v,z_0] \in \mathcal{Z}_{s/2}$ for some $s \in [2,\infty]$.

Lemma 3.2.10. *Let g_i be of order at least $q_i \in \mathbb{N}$. Then*

$$\begin{aligned}&t \mapsto g_i(y_t) &&\in W^{q_i,\infty}([0,T]),\\ &t \mapsto g_i'(y_t)z_t &&\in W^{q_i,s}([0,T]),\\ &t \mapsto g_i''(y_t)(z_t)^2 + g_i'(y_t)z_t^2 &&\in W^{q_i,s/2}([0,T]),\end{aligned}$$

and for $j = 1,\ldots,q_i$,

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}g_i(y)|_t = g_i^{(j)}(t,u_t,y_t,u,y), \tag{3.28}$$

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j}g_i'(y)z|_t = D_{(\tilde{u},\tilde{y},u,y)}g_i^{(j)}(t,u_t,y_t,u,y)(v_t,z_t,v,z), \tag{3.29}$$

$$\begin{aligned}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\left(g_i''(y_t)(z_t)^2 + g_i'(y_t)z_t^2\right)|_t = D^2_{(\tilde{u},\tilde{y},u,y)^2}g_i^{(j)}(t,u_t,y_t,u,y)(v_t,z_t,v,z)^2\\ + D_{(\tilde{y},y)}g_i^{(j)}(t,u_t,y_t,u,y)(z_t^2,z^2).\end{aligned} \tag{3.30}$$

Proof. The result follows from Lemmas 3.2.6, 3.2.7, 3.2.8, by induction on j. Observe in particular that by Definition 3.2.9, formulas (3.28)-(3.30) do not depend on u_t nor v_t for $j = 1,\ldots,q_i - 1$. □

Example 3.2.11. 1. The classical example of a state constraint of order q is $y_t \leq 0$ for all $t \in [0,T]$ where $y_t^{(q)} = u_t$ for a.a. $t \in (0,T)$. This higher-order controlled differential equation can be written as a system of controlled differential equations and the notion of order of state constraints for ODEs applies. Or it is interesting to note that this equation can be reduced to the following scalar integral equation:

$$y_t = \int_0^t \frac{(t-s)^{q-1}}{(q-1)!} u_s \mathrm{d}s, \quad t \in [0,T].$$

Then the constraint $y_t \leq 0$ for all $t \in [0,T]$ gives

$$\begin{aligned} g^{(0)}(t, \tilde{u}, \tilde{y}, u, y) &= \tilde{y}, \\ g^{(j)}(t, \tilde{u}, \tilde{y}, u, y) &= \int_0^t \frac{(t-s)^{q-1-j}}{(q-1-j)!} u_s \mathrm{d}s, \quad j = 1, \dots, q-1, \\ g^{(q)}(t, \tilde{u}, \tilde{y}, u, y) &= \tilde{u}. \end{aligned}$$

Thus we find again that the constraint is of order q.

2. We consider the following variant of the previous example:

$$y_t = \int_0^t \frac{(t-s)^{q-1}}{(q-1)!} f(t,s) u_s \mathrm{d}s, \quad t \in [0,T].$$

If f is not polynomial in t, then this integral equation cannot be in general reduced to a system of ODEs (see Remark 3.2.1.1). And the constraint $y_t \leq 0$ for all $t \in [0,T]$ is still of order q.

3.3 Weak results

3.3.1 A first abstract formulation

The optimal control problem (P) can be rewritten as an abstract optimization problem on (u, y_0). The most naive way to do that is the following equivalent formulation:

$$(P) \qquad \min_{(u,y_0) \in \mathcal{U} \times \mathbb{R}^n} J(u, y_0) \tag{3.31}$$

$$\text{subject to} \qquad g(y[u, y_0]) \in C\left([0,T]; \mathbb{R}^r_-\right), \tag{3.32}$$

$$\Phi(y_0, y[u, y_0]_T) \in K, \tag{3.33}$$

where

$$J(u, y_0) := \int_0^T \ell(u_t, y[u, y_0]_t) \mathrm{d}t + \phi(y_0, y[u, y_0]_T) \tag{3.34}$$

and $\Phi = (\Phi^E, \Phi^I)$, $K = \{0\}_{s_E} \times \mathbb{R}^{s_I}_-$. In order to write optimality conditions for this problem, we first compute its Lagrangian

$$L(u, y_0, \mu, \Psi) := J(u, y_0) + \int_{[0,T]} \mathrm{d}\mu_t g(y[u, y_0]_t) + \Psi \Phi(y_0, y[u, y_0]_T)$$

where $(u, y_0, \mu, \Psi) \in \mathcal{U} \times \mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^{s*}$ (see the beginning of Section 3.2.2). A *Lagrange multiplier* at (u, y_0) in this setting is any (μ, Ψ) such that

$$D_{(u,y_0)} L(u, y_0, \mu, \Psi) \equiv 0, \tag{3.35}$$

$$(\mu, \Psi) \in N_{C([0,T]; \mathbb{R}^r_-) \times K}\left(g(y), \Phi(y_0, y_T)\right), \tag{3.36}$$

where $N_{C([0,T];\mathbb{R}^r_-)\times K}(g(y),\Phi(y_0,y_T))$ is the normal cone to

$$C([0,T];\mathbb{R}^r_-)\times K \text{ at } (g(y),\Phi(y_0,y_T)).$$

We have the following characterization:

$$(\mu,\Psi)\in N_{C([0,T];\mathbb{R}^r_-)\times K}(g(y),\Phi(y_0,y_T))$$

iff

$$\begin{aligned}&g_i(y)\le 0,\quad \mu_i\ge 0,\quad \int_0^T g_i(y_t)\mathrm{d}\mu_{i,t}=0,\quad i=1,\dots,r,\\ &\Psi\in N_K(\Phi(y_0,y_T))\quad \text{(see (3.9))}.\end{aligned}$$

Definition (3.35)-(3.36) has to be compared to Definition 3.2.3:

Lemma 3.3.1. *The couple* (μ,Ψ) *is a Lagrange multiplier of the abstract problem* (3.31)-(3.33) *at* $(\bar u,\bar y_0)$ *iff* (μ,Ψ,p) *is a Lagrange multiplier of the optimal control problem* (3.1)-(3.5) *associated with* $(\bar u, y[\bar u,\bar y_0])$*, where* p *is the unique solution of* (3.11)*.*

Proof. Using the Hamiltonian (3.7) and the end points Lagrangian (3.8), we have

$$\begin{aligned}L(u,y_0,\mu,\Psi)=\int_0^T H[p](t,u_t,y_t)\mathrm{d}t+\int_{[0,T]}\mathrm{d}\mu_t g(y_t)+\Phi[\Psi](y_0,y_T)\\ -\int_0^T\left(p_t f(t,t,u_t,y_t)+\int_t^T p_s D_\tau f(s,t,u_t,y_t)\mathrm{d}s\right)\mathrm{d}t.\end{aligned}$$

for $y=y[u,y_0]$ and any $p\in\mathcal{P}$. Moreover

$$\begin{aligned}&\int_0^T\left(p_t f(t,t,u_t,y_t)+\int_t^T p_s D_\tau f(s,t,u_t,y_t)\mathrm{d}s\right)\mathrm{d}t\\ &\qquad=\int_0^T p_t\left(f(t,t,u_t,y_t)+\int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s\right)\mathrm{d}t\\ &\qquad\qquad=\int_{[0,T]}p_t\dot y_t\mathrm{d}t=-\int_{[0,T]}\mathrm{d}p_t y_t+p_T y_T-p_0y_0\end{aligned}$$

by the formula of integration by parts (3.89) of the Appendix 3.A.1. Then

$$\begin{aligned}L(u,y_0,\mu,\Psi)=\int_0^T H[p](t,u_t,y_t)\mathrm{d}t+\int_{[0,T]}(\mathrm{d}p_t y_t+\mathrm{d}\mu_t g(y_t))\\ +p_0y_0-p_Ty_T+\Phi[\Psi](y_0,y_T)\end{aligned}$$

for any $p\in\mathcal{P}$. We fix $(\bar u,\bar y_0,\mu,\Psi)$, we differentiate L w.r.t. (u,y_0) at this point, and we choose p as the unique solution of (3.11). Then

$$\begin{aligned}D_{(u,y_0)}L(\bar u,\bar y_0,\mu,\Psi)(v,z_0)=\int_0^T D_u H[p](t,\bar u_t,\bar y_t)v_t\mathrm{d}t\\ +\left(p_0+D_{y_1}\Phi[\Psi](\bar y_0,\bar y_T)\right)z_0\end{aligned}$$

for all $(v,z_0)\in\mathcal{U}\times\mathbb{R}^n$. It follows that (3.35) is equivalent to (3.12) and (3.15). And it is obvious that (3.36) is equivalent to (3.13)-(3.14). □

Second we need a qualification condition.

Definition 3.3.2. We say that $(\bar{u}, \bar{y})$ is *qualified* iff

(i) $\left\{ \begin{array}{ccc} (v, z_0) & \mapsto & D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, z[v, z_0]_T) \\ \mathcal{U} \times \mathbb{R}^n & \to & \mathbb{R}^{s_E} \end{array} \right.$ is onto,

(ii) there exists $(\bar{v}, \bar{z}_0) \in \mathcal{U} \times \mathbb{R}^n$ such that, with $\bar{z} = z[\bar{v}, \bar{z}_0]$,

$$\left\{ \begin{array}{ll} D\Phi^E(\bar{y}_0, \bar{y}_T)(\bar{z}_0, \bar{z}_T) = 0, & \\ D\Phi_i^I(\bar{y}_0, \bar{y}_T)(\bar{z}_0, \bar{z}_T) < 0, & i \in \left\{ i \, : \, \Phi_i^I(\bar{y}_0, \bar{y}_T) = 0 \right\}, \\ g_i'(\bar{y}_t)\bar{z}_t < 0 \text{ on } \left\{ t \, : \, g_i(\bar{y}_t) = 0 \right\}, & i = 1, \ldots, r. \end{array} \right.$$

Remark 3.3.3. 1. This condition is equivalent to Robinson's constraint qualification (introduced in [80], Definition 2) for the abstract problem (3.31)-(3.33) at $(\bar{u}, \bar{y}_0)$; see the discussion that follows Definition 3.4 and Definition 3.5 in [58] for a proof of the equivalence.

2. It is sometimes possible to give optimality conditions without qualification condition by considering an auxiliary optimization problem (see e.g. the proof of Theorem 3.50 in [25]). Nevertheless, observe that if $(\bar{u}, \bar{y})$ is feasible but not qualified because (i) does not hold, then there exists a *singular Langrange multiplier* of the form $(0, \Phi^E, 0)$. One can see that second-order necessary conditions become pointless since $-(0, \Phi^E, 0)$ is a singular Lagrange multiplier too. In this perspective, we only consider qualified solutions.

Finally we derive the following first-order necessary optimality conditions:

Theorem 3.3.4. *Let $(\bar{u}, \bar{y})$ be a qualified local solution of (P). Then the set of associated Lagrange multipliers is nonempty, convex and weakly $*$ compact.*

Proof. Since the abstract problem (3.31)-(3.33) is qualified, we get the result for the set $\{(\mu, \Psi)\}$ of Lagrange multipliers in this setting (Theorem 4.1 in [92]). We conclude with Lemma 3.3.1 and the fact that

$$\begin{array}{c} \mathcal{M} \times \mathbb{R}^{s*} \longrightarrow \mathcal{M} \times \mathbb{R}^{s*} \times \mathcal{P} \\ (\mu, \Psi) \longmapsto (\mu, \Psi, p) \end{array}$$

is affine continuous (it is obvious from the proof of Lemma 3.2.2). □

We will prove a stronger result in Section 3.4, relying on another abstract formulation, the so-called *reduced problem*. The main motivation for the reduced problem, as mentioned in the introduction, is actually to satisfy an *extended polyhedricity condition* (see Definition 3.52 in [25]), in order to easily get second-order necessary conditions (see Remark 3.47 in the same reference).

3.3.2 The reduced problem

In the sequel we fix a feasible trajectory $(\bar{u}, \bar{y})$, i.e. which satisfies (3.2)-(3.5), and denote by Λ the set of associated Lagrange multipliers (Definition 3.2.3). We need some definitions:

Definition 3.3.5. An *arc* is a maximal interval, relatively open in $[0,T]$, denoted by $]\tau_1,\tau_2[$, such that the set of active running state constraints at time t is constant for all $t \in]\tau_1,\tau_2[$. It includes intervals of the form $[0,\tau[$ or $]\tau,T]$. If τ does not belong to any arc, we say that τ is a *junction time.*

Consider an arc $]\tau_1,\tau_2[$. It is a *boundary arc* for the constraint g_i if the latter is active on $]\tau_1,\tau_2[$; otherwise it is an *interior arc* for g_i.

Consider an interior arc $]\tau_1,\tau_2[$ for g_i. If $g_i(\tau_2)=0$, then τ_2 is an *entry point* for g_i; if $g_i(\tau_1)=0$, then τ_1 is an *exit point* for g_i. If τ is an entry point and an exit point, then it is a *touch point* for g_i.

Consider a touch point τ for g_i. We say that τ is *reducible* iff $\frac{\mathrm{d}^2}{\mathrm{d}t^2}g_i(\bar y_t)$, defined in a weak sense, is a function for t close to τ, continuous at τ, and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}g_i(\bar y_t)|_{t=\tau}<0.$$

Remark 3.3.6. Let g_i be of order at least 2 and τ be a touch point for g_i. By Lemma 3.2.10, τ is reducible iff $t\mapsto g_i^{(2)}(t,\bar u_t,\bar y_t,\bar u,\bar y)$ is continuous at τ and $g_i^{(2)}(\tau,\bar u_\tau,\bar y_\tau,\bar u,\bar y)<0$. Note that the continuity holds if $\bar u$ is continuous at τ or if g_i is of order at least 3.

The interest of reducibility will appear with the next lemma. For $\tau\in[0,T]$, $\varepsilon>0$ (to be fixed) and any function $x\colon [0,T]\to\mathbb{R}$, $x\in W^{2,\infty}$, we define $\mu_\tau(x)$ by

$$\mu_\tau(x):=\max\left\{x_t\ :\ t\in[\tau-\varepsilon,\tau+\varepsilon]\cap[0,T]\right\}.$$

Thus we get a functional $\mu_\tau\colon W^{2,\infty}([0,T])\to\mathbb{R}$.

Lemma 3.3.7. *Let g_i be of order at least 2 (i.e. $D_{\tilde u}g_i^{(1)}\equiv 0$) and hence by Lemma 3.2.10 $g_i(\bar y)\in W^{2,\infty}$. Let τ be a reducible touch point for g_i. Then for $\varepsilon>0$ small enough, μ_τ is C^1 in a neighbourhood of $g_i(\bar y)$ and twice Fréchet differentiable at $g_i(\bar y)$, with first and second derivatives at $g_i(\bar y)$ given by*

$$D\mu_\tau(g_i(\bar y))x=x_\tau, \tag{3.37}$$

$$D^2\mu_\tau(g_i(\bar y))(x)^2=-\frac{\left(\dfrac{\mathrm{d}}{\mathrm{d}t}x_t|_\tau\right)^2}{\dfrac{\mathrm{d}^2}{\mathrm{d}t^2}g_i(\bar y_t)|_\tau}, \tag{3.38}$$

for any $x\in W^{2,\infty}([0,T])$.

Proof. We apply Lemma 23 of [20] to $g_i(\bar y)$, which belongs to $W^{2,\infty}([0,T])$ and satisfies the required hypotheses at τ by definition of a reducible touch point. ∎

Remark 3.3.8. We can write (3.37) and (3.38) for $x=g_i'(\bar y)z[v,z_0]$ or $x=g_i''(\bar y)(z[v,z_0])^2+g_i'(\bar y)z^2[v,z_0]$, $(v,z_0)\in\mathcal{U}\times\mathbb{R}^n$, since by Lemma 3.2.10 they belong to $W^{2,\infty}([0,T])$. Moreover, we have

$$D^2\mu_\tau(g_i(\bar y))(g_i'(\bar y)z)^2=-\frac{\left(D_{(\tilde y,u,y)}g_i^{(1)}(\tau,\bar y_\tau,\bar u,\bar y)(z_\tau,v,z)\right)^2}{g_i^{(2)}(\tau,\bar u_\tau,\bar y_\tau,\bar u,\bar y)} \tag{3.39}$$

for $z=z[v,z_0]$, $(v,z_0)\in\mathcal{U}\times\mathbb{R}^n$.

In view of these results we distinguish running state constraints of order 1. Without loss of generality, we suppose that

– g_i is of order 1 for $i = 1, \dots, r_1$,

– g_i is of order at least 2 for $i = r_1 + 1, \dots, r$,

where $0 \leq r_1 \leq r$. We make now the following assumption:

(A1) There are finitely many junction times, and for $i = r_1 + 1, \dots, r$ all touch points for g_i are reducible.

For $i = 1, \dots, r_1$ we consider the contact sets of the constraints

$$\Delta_i := \{t \in [0, T] \, : \, g_i(\bar{y}_t) = 0\}.$$

For $i = r_1 + 1, \dots, r$ we remove the touch points from the contact sets:

$$\begin{aligned} \mathcal{T}_i &:= \text{the set of (reducible) touch points for } g_i, \\ \Delta_i &:= \{t \in [0, T] \, : \, g_i(\bar{y}_t) = 0\} \setminus \mathcal{T}_i. \end{aligned} \tag{3.40}$$

For $i = 1, \dots, r$ and $\varepsilon \geq 0$ we denote

$$\Delta_i^\varepsilon := \{t \in [0, T] \, : \, \operatorname{dist}(t, \Delta_i) \leq \varepsilon\}.$$

Assumption (A1) implies that Δ_i^ε has finitely many connected components for any $\varepsilon \geq 0$ ($1 \leq i \leq r$) and that $\mathcal{T}_i$ is finite ($r_1 < i \leq r$). Let $N := \sum_{r_1 < i \leq r} |\mathcal{T}_i|$.

Now we fix $\varepsilon > 0$ small enough (so that Lemma 3.3.7 holds) and we define

$$\begin{aligned} G_1(u, y_0) &:= \left(g_i\left(y[u, y_0]\right)|_{\Delta_i^\varepsilon}\right)_{1 \leq i \leq r}, & K_1 &:= \prod_{i=1}^r C\left(\Delta_i^\varepsilon, \mathbb{R}_-\right), \\ G_2(u, y_0) &:= \left(\mu_\tau\left(g_i\left(y[u, y_0]\right)\right)\right)_{\tau \in \mathcal{T}_i,\ r_1 < i \leq r}, & K_2 &:= \mathbb{R}_-^N, \\ G_3(u, y_0) &:= \Phi\left(y_0, y[u, y_0]_T\right), & K_3 &:= K. \end{aligned}$$

Note that Lemma 3.3.7 does not enable us to consider touch points for constraints of order 1 in G_2, since we want the later to be regular enough. This is not a problem, we treat them with the boundary arcs in G_1 and we will see that an extended polyhedricity condition (Lemma 3.4.9) is satisfied.

Recall that J has been defined by (3.34); the **reduced problem** is the following abstract optimization problem:

$$(P_R) \qquad \min_{(u, y_0) \in \mathcal{U} \times \mathbb{R}^n} J(u, y_0), \quad \text{subject to} \quad \left\{ \begin{array}{l} G_1(u, y_0) \in K_1 \\ G_2(u, y_0) \in K_2 \\ G_3(u, y_0) \in K_3 \end{array} \right. .$$

Remark 3.3.9. We had fixed $(\bar{u}, \bar{y})$ as a feasible trajectory; then $(\bar{u}, \bar{y}_0)$ is feasible for (P_R). Moreover, $(\bar{u}, \bar{y})$ is a local solution of (P) iff $(\bar{u}, \bar{y}_0)$ is a local solution of (P_R), and the qualification condition at $(\bar{u}, \bar{y})$ (Definition 3.3.2) is equivalent to Robinson's constraints qualification for (P_R) at $(\bar{u}, \bar{y}_0)$ (using Lemma 3.3.7).

Thus it is of interest for us to write optimality conditions for (P_R).

3.3.3 Optimality conditions for the reduced problem

The *Lagrangian* of (P_R) is

$$L_R(u, y_0, \rho, \nu, \Psi) := J(u, y_0) + \sum_{1 \le i \le r} \int_{\Delta_i^\varepsilon} g_i(y[u, y_0]_t) \mathrm{d}\rho_{i,t} \\ + \sum_{\substack{\tau \in \mathcal{T}_i \\ r_1 < i \le r}} \nu_{i,\tau} \mu_\tau \left(g_i \left(y[u, y_0] \right) \right) + \Psi \Phi(y_0, y[u, y_0]_T) \quad (3.41)$$

where $\quad u \in \mathcal{U}, \quad y_0 \in \mathbb{R}^n, \quad \rho \in \prod_{i=1}^{r} \mathcal{M}\left(\Delta_i^\varepsilon\right), \quad \nu \in \mathbb{R}^{N*}, \quad \Psi \in \mathbb{R}^{s*}.$

As before, a measure on a closed interval is denoted by $\mathrm{d}\mu$ and is identified with the derivative of a function of bounded variations which is null on the right of the interval.

A *Lagrange multiplier* of (P_R) at $(\bar{u}, \bar{y}_0)$ is any (ρ, ν, Ψ) such that

$$D_{(u,y_0)} L_R(\bar{u}, \bar{y}_0, \rho, \nu, \Psi) = 0, \quad (3.42)$$

$$\rho_i \ge 0, \; g_i(\bar{y})|_{\Delta_i^\varepsilon} \le 0, \; \int_{\Delta_i^\varepsilon} g_i(\bar{y}_t) \mathrm{d}\rho_{i,t} = 0, \quad i = 1, \dots, r, \quad (3.43)$$

$$\nu_{i,\tau} \ge 0, \; \mu_\tau \left(g_i(\bar{y}) \right) \le 0, \; \nu_{i,\tau} \mu_\tau \left(g_i(\bar{y}) \right) = 0, \quad \tau \in \mathcal{T}_i, \; i = r_1 + 1, \dots, r, \quad (3.44)$$

$$\Psi \in N_K \left(\Phi(\bar{y}_0, \bar{y}_T) \right). \quad (3.45)$$

We denote by Λ_R the set of Lagrange multipliers of (P_R) at $(\bar{u}, \bar{y}_0)$. The first-order necessary conditions for (P_R) are the same as in Theorem 3.3.4:

Lemma 3.3.10. *Let $(\bar{u}, \bar{y}_0)$ be a qualified local solution of (P_R). Then Λ_R is nonempty, convex and weakly $*$ compact.*

Given $(\rho, \nu) \in \prod_{i=1}^r \mathcal{M}\left(\Delta_i^\varepsilon\right) \times \mathbb{R}^{N*}$, we define $\mu \in \mathcal{M}$ by

$$\mu_i := \begin{cases} \rho_i & \text{on } \Delta_i^\varepsilon, \quad i = 1, \dots, r, \\ \sum_{\tau \in \mathcal{T}_i} \nu_{i,\tau} \delta_\tau & \text{elsewhere}, \quad i = r_1 + 1, \dots, r. \end{cases} \quad (3.46)$$

Conversely, given $\mu \in \mathcal{M}$, we define $(\rho, \nu) \in \prod_{i=1}^r \mathcal{M}\left(\Delta_i^\varepsilon\right) \times \mathbb{R}^{N*}$ by

$$\begin{cases} \rho_i := \mu_i|_{\Delta_i^\varepsilon} & i = 1, \dots, r, \\ \nu_{i,\tau} := \mu_i(\{\tau\}) & \tau \in \mathcal{T}_i, \quad i = r_1 + 1, \dots, r. \end{cases} \quad (3.47)$$

In the sequel we use these definitions to identify (ρ, ν) and μ, and we denote

$$[\mu_{i,\tau}] := \mu_i(\{\tau\}). \quad (3.48)$$

Recall that Λ is the set of Lagrange multipliers associated with $(\bar{u}, \bar{y})$ (Definition 3.2.3). We have a result similar to Lemma 3.3.1:

Lemma 3.3.11. *The triple $(\rho, \nu, \Psi) \in \Lambda_R$ iff $(\mu, \Psi, p) \in \Lambda$, with p the unique solution of* (3.11).

Proof. With the identification between (ρ, ν) and μ given by (3.46) and (3.47), it is clear that (3.43)-(3.44) are equivalent to (3.13). Let these relations be satisfied by (ρ, ν, Ψ) and (μ, Ψ). Then in particular

$$\begin{array}{ll} \operatorname{supp}(\mu_i) = \operatorname{supp}(\rho_i) \subset \Delta_i & i = 1, \dots, r_1, \\ \operatorname{supp}(\mu_i) = \operatorname{supp}(\rho_i) \cup \operatorname{supp}(\sum \nu_{i,\tau}\delta_\tau) \subset \Delta_i \cup \mathcal{T}_i & i = r_1 + 1, \dots, r. \end{array} \tag{3.49}$$

We claim that in this case (3.42) is equivalent to (3.12) and (3.15). Indeed, as in the proof of Lemma 3.3.1, we have

$$\begin{aligned} L_R(u, y_0, \rho, \nu, \Psi) = & \int_{[0,T]} (H[p](t, u_t, y_t)\mathrm{d}t + \mathrm{d}p_t y_t) + p_0 y_0 - p_T y_T \\ & + \sum_{1 \le i \le r} \int_{\Delta_i} g_i(y_t)\mathrm{d}\mu_{i,t} + \sum_{\substack{\tau \in \mathcal{T}_i \\ r_1 < i \le r}} [\mu_{i,\tau}]\mu_\tau\left(g_i(y)\right) + \Phi[\Psi](y_0, y_T) \end{aligned} \tag{3.50}$$

for any $p \in \mathcal{P}$ and $y = y[u, y_0]$. Let us differentiate (say for $i > r_1$)

$$\int_{\Delta_i} g_i(y_t)\mathrm{d}\mu_{i,t} + \sum_{\tau \in \mathcal{T}_i} [\mu_{i,\tau}]\mu_\tau\left(g_i(y)\right) \tag{3.51}$$

w.r.t. (u, y_0) at $(\bar{u}, \bar{y}_0)$ in the direction (v, z_0) and use (3.37), (3.48), (3.49); we get

$$\int_{\Delta_i} g_i'(\bar{y}_t)z_t\mathrm{d}\mu_{i,t} + \sum_{\tau \in \mathcal{T}_i} [\mu_{i,\tau}]D\mu_\tau\left(g_i(\bar{y})\right)\left(g_i'(\bar{y})z\right) = \int_{[0,T]} g_i'(\bar{y}_t)z_t\mathrm{d}\mu_{i,t}$$

where $z = z[v, z_0]$. Let us now differentiate similarly the whole expression (3.50) of L_R; we get

$$\begin{aligned} \int_0^T D_u H[p](t, \bar{u}_t, \bar{y}_t)v_t\mathrm{d}t + & \int_{[0,T]} (D_y H[p](t, \bar{u}_t, \bar{y}_t)\mathrm{d}t + \mathrm{d}p_t + \mathrm{d}\mu_t g'(\bar{y}_t))z_t \\ & + (p_0 + D_{y_1}\Phi[\Psi](\bar{y}_0, \bar{y}_T))z_0 + (-p_T + D_{y_2}\Phi[\Psi](\bar{y}_0, \bar{y}_T))z_T. \end{aligned} \tag{3.52}$$

Fixing p as the unique solution of (3.11) in (3.52) gives

$$D_{(u,y_0)}L_R(\bar{u}, \bar{y}_0, \rho, \nu, \Psi)(v, z_0) = \int_0^T D_u H[p](t, \bar{u}_t, \bar{y}_t)v_t\mathrm{d}t + (p_0 + D_{y_1}\Phi[\Psi](\bar{y}_0, \bar{y}_T))z_0.$$

It is now clear that (3.42) is equivalent to (3.12) and (3.15). □

For the second-order optimality conditions, we need to evaluate the Hessian of L_R. For $\lambda = (\mu, \Psi, p) \in \Lambda$, $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$ and $z = z[v, z_0] \in \mathcal{Y}$, we denote

$$\begin{aligned} \Omega[\lambda](v, z_0) := & \int_0^T D^2_{(u,y)^2}H[p](t, \bar{u}_t, \bar{y}_t)(v_t, z_t)^2\mathrm{d}t + D^2\Phi[\Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T)^2 \\ & + \sum_{1 \le i \le r} \int_{\Delta_i} g_i''(\bar{y}_t)(z_t)^2\mathrm{d}\mu_{i,t} \\ & + \sum_{\substack{\tau \in \mathcal{T}_i \\ r_1 < i \le r}} [\mu_{i,\tau}]\left[g_i''(\bar{y}_\tau)(z_\tau)^2 + D^2\mu_\tau\left(g_i(\bar{y})\right)\left(g_i'(\bar{y})z\right)^2\right]. \end{aligned} \tag{3.53}$$

In view of (3.39) and (3.49), we could also write

$$\Omega[\lambda](v,z_0) = \int_0^T D^2_{(u,y)^2} H[p](t,\bar{u}_t,\bar{y}_t)(v_t,z_t)^2 \mathrm{d}t + D^2\Phi[\Psi](\bar{y}_0,\bar{y}_T)(z_0,z_T)^2$$
$$+ \int_{[0,T]} \mathrm{d}\mu_t g''(\bar{y}_t)(z_t)^2 - \sum_{\substack{\tau\in\mathcal{T}_i \\ r_1<i\le r}} [\mu_{i,\tau}] \frac{\left(D_{(\tilde{y},u,y)} g_i^{(1)}(\tau,\bar{y}_\tau,\bar{u},\bar{y})(z_\tau,v,z)\right)^2}{g_i^{(2)}(\tau,\bar{u}_\tau,\bar{y}_\tau,\bar{u},\bar{y})}. \quad (3.54)$$

Lemma 3.3.12. *Let $(\rho,\nu,\Psi)\in\Lambda_R$. Let $(\mu,\Psi,p)\in\Lambda$ be given by Lemma 3.3.11 and denoted by λ. Then for all $(v,z_0)\in\mathcal{U}\times\mathbb{R}^n$,*

$$D^2_{(u,y_0)^2} L_R(\bar{u},\bar{y}_0,\rho,\nu,\Psi)(v,z_0)^2 = \Omega[\lambda](v,z_0).$$

Proof. We will use (3.50) and (3.51) from the previous proof. First we differentiate (3.51) twice w.r.t. (u,y_0) at $(\bar{u},\bar{y}_0)$ in the direction (v,z_0). Let $z=z[v,z_0]$ and $z^2=z^2[v,z_0]$, defined by (3.19); we get for $i>r_1$

$$\int_{\Delta_i}\left(g_i''(\bar{y}_t)(z_t)^2 + g_i'(\bar{y}_t)z_t^2\right)\mathrm{d}\mu_{i,t}$$
$$+\sum_{\tau\in\mathcal{T}_i}[\mu_{i,\tau}]\left[D^2\mu_\tau\left(g_i(\bar{y})\right)\left(g_i'(\bar{y})z\right)^2 + D\mu_\tau\left(g_i(\bar{y})\right)\left(g_i''(\bar{y})(z)^2+g_i'(\bar{y})z^2\right)\right]$$
$$=\int_{\Delta_i} g_i''(\bar{y}_t)(z_t)^2\mathrm{d}\mu_{i,t} + \int_{[0,T]} g_i'(\bar{y}_t)z_t^2\mathrm{d}\mu_{i,t}$$
$$+\sum_{\tau\in\mathcal{T}_i}[\mu_{i,\tau}]\left[D^2\mu_\tau\left(g_i(\bar{y})\right)\left(g_i'(\bar{y})z\right)^2 + g_i''(\bar{y}_\tau)(z_\tau)^2\right],$$

where we have used Remark 3.3.8, (3.37) and (3.49). Second we differentiate L_R twice using (3.50) and then we fix p as the unique solution of (3.11). The result follows as in the proof of Lemma 3.3.11. □

Suppose that $\Lambda\neq\emptyset$ and let $\bar{\lambda}=(\bar{\mu},\bar{\Psi},\bar{p})\in\Lambda$. We define the *critical L^2 cone* as the set C_2 of $(v,z_0)\in\mathcal{V}_2\times\mathbb{R}^n$ such that

$$\begin{cases} g_i'(\bar{y})z\le 0 & \text{on } \Delta_i, \\ g_i'(\bar{y})z=0 & \text{on supp}\,(\bar{\mu}_i)\cap\Delta_i, \end{cases} \quad i=1,\dots,r, \quad (3.55)$$

$$\begin{cases} g_i'(\bar{y}_\tau)z_\tau\le 0, \\ [\bar{\mu}_{i,\tau}]\,g_i'(\bar{y}_\tau)z_\tau=0, \end{cases} \quad \tau\in\mathcal{T}_i,\ i=r_1+1,\dots,r, \quad (3.56)$$

$$\begin{cases} D\Phi(\bar{y}_0,\bar{y}_T)(z_0,z_T)\in T_K\left(\Phi(\bar{y}_0,\bar{y}_T)\right), \\ \bar{\Psi}D\Phi(\bar{y}_0,\bar{y}_T)(z_0,z_T)=0, \end{cases} \quad (3.57)$$

where $z=z[v,z_0]\in\mathcal{Z}_2$. Then the *critical cone* for (P_R) (see Proposition 3.10 in [25]) is the set

$$C_\infty := C_2\cap\left(\mathcal{U}\times\mathbb{R}^n\right),$$

and the *cone of radial critical directions* for (P_R) (see Definition 3.52 in [25]) is the set

$$C_\infty^R := \left\{(v,z_0)\in C_\infty \,:\, \exists\bar{\sigma}>0 \,:\, g_i(\bar{y})+\bar{\sigma}g_i'(\bar{y})z\le 0 \text{ on } \Delta_i^\varepsilon,\ i=1,\dots,r\right\},$$

where $z=z[v,z_0]\in\mathcal{Y}$. These three cones do not depend on the choice of $\bar{\lambda}$. In view of Lemma 3.3.12, the second-order necessary conditions for (P_R) can be written as follows:

Lemma 3.3.13. *Let $(\bar{u}, \bar{y}_0)$ be a qualified local solution of (P_R). Then for any $(v, z_0) \in C^R_\infty$, there exists $\lambda \in \Lambda$ such that*

$$\Omega[\lambda](v, z_0) \geq 0.$$

Proof. Corollary 5.1 in [58]. □

3.4 Strong results

Recall that $(\bar{u}, \bar{y})$ is a feasible trajectory that has been fixed to define the reduced problem at the beginning of Section 3.3.2.

3.4.1 Extra assumptions and consequences

We were so far under the assumptions (A0)-(A1). We make now some extra assumptions, which will imply a partial qualification of the running state constraints, as well as the density of C^R_∞ in a larger critical cone.

(A2) Each running state constraint $g_i, i = 1, \dots, r$ is of finite order q_i.

Notations Given a subset $J \subset \{1, \dots, r\}$, say $J = \{i_1 < \dots < i_l\}$, we define

$$G^{(q)}_J : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{Y} \to \mathbb{R}^{|J|} \quad \text{by}$$
$$G^{(q)}_J(t, \tilde{u}, \tilde{y}, u, y) := \begin{pmatrix} \bar{g}^{(q_{i_1})}_{i_1}(t, \tilde{u}, \tilde{y}, u, y) \\ \vdots \\ \bar{g}^{(q_{i_l})}_{i_l}(t, \tilde{u}, \tilde{y}, u, y)T \end{pmatrix}.$$

For $\varepsilon_0 \geq 0$ and $t \in [0, T]$, let

$$I^{\varepsilon_0}_t := \{1 \leq i \leq r \,:\, t \in \Delta^{\varepsilon_0}_i\},$$
$$M^{\varepsilon_0}_t := D_{\tilde{u}} G^{(q)}_{I^{\varepsilon_0}_t}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) \in M_{|I^{\varepsilon_0}_t|, m}(\mathbb{R}).$$

(A3) There exists $\varepsilon_0, \gamma > 0$ such that, for all $t \in [0, T]$,

$$|(M^{\varepsilon_0}_t)^* \xi| \geq \gamma |\xi| \quad \forall \xi \in \mathbb{R}^{|I^{\varepsilon_0}_t|}.$$

(A4) The initial condition satisfies $g(\bar{y}_0) < 0$ and the final time T is not an entry point (i.e. there exists $\tau < T$ such that the set I^0_t of active constraints at time t is constant for $t \in]\tau, T]$).

Remark 3.4.1.
1. We do not assume that $\bar{u}$ is continuous, as was done in [21].
2. Assumption (A3) says that $M^{\varepsilon_0}_t$ is onto, uniformly w.r.t. t. Note that each constraint is considered only in a neighbourhood of its contact set. Note also that in the case of one running state constraint ($r = 1$) of order q and if $\bar{u}$ is continuous, assumption (A3) is equivalent to
$$\frac{\partial g^{(q)}}{\partial \tilde{u}}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) \neq 0 \quad \forall t \in \Delta.$$
See Appendix 3.A.2 for the example 3.A.6, where this assumption is discussed.

3. Recall that ε has been fixed to define the reduced problem. Without loss of generality we suppose that $2\varepsilon_0 < \min\{|\tau - \tau'| \, : \, \tau, \tau' \text{ distinct junction times}\}$ and $\varepsilon < \varepsilon_0 < \min\{\tau \, : \, \tau \text{ junction times}\}$. We omit it in the notation $M_t^{\varepsilon_0}$.
4. In some cases, we can treat the case where T is an entry point, say for the constraint g_i:
   – if $1 \leq i \leq r_1$ (i.e. if $q_i = 1$), then what follows works similarly.
   – if $r_1 < i \leq r$ (i.e. if $q_i > 1$) and $\frac{\mathrm{d}}{\mathrm{d}t} g_i(\bar{y}_t)|_{t=T} > 0$, then we can replace in the reduced problem the running state constraint $g_i(y[u, y_0])|_{[T-\varepsilon, T]} \leq 0$ by the final state constraint $g_i(y[u, y_0]_T) \leq 0$.
5. By assumption (A1), we can write

$$[0, T] = J_0 \cup \cdots \cup J_\kappa \tag{3.58}$$

where J_l $(l = 0, \ldots, \kappa)$ are the maximal intervals in $[0, T]$ such that $I_t^{\varepsilon_0}$ is constant (say equal to I_l) for $t \in J_l$. We order $J_0, \ldots, J_\kappa$ in $[0, T]$. Observe that for any $l \geq 1$, $\overline{J_{l-1}} \cap \overline{J_l} = \{\tau \pm \varepsilon_0\}$ with τ a junction time.

For $s \in [1, \infty]$, we denote

$$W^{(q),s}([0, T]) := \prod_{i=1}^{r} W^{q_i,s}([0, T]), \quad W^{(q),s}(\Delta^\varepsilon) := \prod_{i=1}^{r} W^{q_i,s}(\Delta_i^\varepsilon),$$

and for $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix} \in W^{(q),s}([0, T])$, $\varphi|_{\Delta^\varepsilon} := \begin{pmatrix} \varphi_1|_{\Delta_1^\varepsilon} \\ \vdots \\ \varphi_r|_{\Delta_r^\varepsilon} \end{pmatrix} \in W^{(q),s}(\Delta^\varepsilon)$.

Using Lemma 3.2.10 we define, for $s \in [1, \infty]$ and $z_0 \in \mathbb{R}^n$,

$$\begin{aligned} \mathcal{A}_{s,z_0} \colon \mathcal{V}_s &\longrightarrow W^{(q),s}([0, T]) \\ v &\longmapsto g'(\bar{y}) z[v, z_0]. \end{aligned}$$

We give now the statement of a lemma in two parts, which will be of great interest for us (particularly in Section 3.4.3). The proof is technical and can be skipped at a first reading. It is given in the next section.

Lemma 3.4.2. *a) Let $s \in [1, \infty]$ and $z_0 \in \mathbb{R}^n$. Let $\bar{b} \in W^{(q),s}(\Delta^\varepsilon)$. Then there exists $v \in \mathcal{V}_s$ such that*

$$(\mathcal{A}_{s,z_0} v)\,|_{\Delta^\varepsilon} = \bar{b}.$$

b) Let $z_0 \in \mathbb{R}^n$. Let $(\bar{b}, \bar{v}) \in W^{(q),2}(\Delta^\varepsilon) \times \mathcal{V}_2$ be such that

$$(\mathcal{A}_{2,z_0} \bar{v})\,|_{\Delta^\varepsilon} = \bar{b}.$$

Let $b^k \in W^{(q),\infty}(\Delta^\varepsilon)$, $k \in \mathbb{N}$, be such that $b^k \xrightarrow{W^{(q),2}(\Delta^\varepsilon)} \bar{b}$. Then there exists $v^k \in \mathcal{U}$, $k \in \mathbb{N}$, such that $v^k \xrightarrow{L^2} \bar{v}$ and

$$\left(\mathcal{A}_{\infty,z_0} v^k\right)|_{\Delta^\varepsilon} = b^k.$$

3.4.2 A technical proof

In this section we prove Lemma 3.4.2. The proofs of a) and b) are very similar; in both cases we proceed in $\kappa+1$ steps using the decomposition (3.58) of $[0,T]$. At each step, we will use the following two lemmas, proved in Appendixes 3.A.3 and 3.A.2, respectively.

The first one uses only assumption (A1) and the definitions that follow.

Lemma 3.4.3. *Let $t_0 := \tau \pm \varepsilon_0$ where τ is a junction time.*

a) Let $s \in [1,\infty]$ and $z_0 \in \mathbb{R}^n$. Let $(\bar{b}, v) \in W^{(q),s}(\Delta^\varepsilon) \times \mathcal{V}_s$ be such that

$$(\mathcal{A}_{s,z_0} v)\,|_{\Delta^\varepsilon} = \bar{b} \text{ on } [0,t_0].$$

Then we can extend $\bar{b}$ to $\tilde{b} \in W^{(q),s}([0,T])$ in such a way that

$$\tilde{b} = \mathcal{A}_{s,z_0} v \text{ on } [0,t_0]. \tag{3.59}$$

b) Let $z_0 \in \mathbb{R}^n$. Let $(\bar{b}, \bar{v}) \in W^{(q),2}(\Delta^\varepsilon) \times \mathcal{V}_2$ be such that

$$(\mathcal{A}_{2,z_0} \bar{v})\,|_{\Delta^\varepsilon} = \bar{b}.$$

Let $(b^k, v^k) \in W^{(q),\infty}(\Delta^\varepsilon) \times \mathcal{U}$, $k \in \mathbb{N}$, be such that

$$\begin{aligned} &(b^k, v^k) \xrightarrow{W^{(q),2} \times L^2} (\bar{b}, \bar{v}), \text{ and} \\ &\left(\mathcal{A}_{\infty,z_0} v^k\right)|_{\Delta^\varepsilon} = b^k \text{ on } [0,t_0]. \end{aligned}$$

Then we can extend b^k to $\tilde{b}^k \in W^{(q),\infty}([0,T])$, $k \in \mathbb{N}$, in such a way that

$$\begin{aligned} &\tilde{b}^k \xrightarrow{W^{(q),2}([0,T])} \mathcal{A}_{2,z_0} \bar{v}, \text{ and} \\ &\tilde{b}^k = \mathcal{A}_{\infty,z_0} v^k \text{ on } [0,t_0]. \end{aligned}$$

The second lemma relies on assumption (A3).

Lemma 3.4.4. *Let $s \in [1,\infty]$ and $z_0 \in \mathbb{R}^n$. Let l be such that $I_l \neq \emptyset$. For $t \in J_l$, we denote*

$$\begin{cases} M_t := D_{\tilde{u}} G_{I_l}^{(q)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) \in \mathcal{L}\left(\mathbb{R}^m, \mathbb{R}^{|I_l|}\right), \\ N_t := D_{(\tilde{y},u,y)} G_{I_l}^{(q)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) \in \mathcal{L}\left(\mathbb{R}^{n*} \times \mathcal{U}^* \times \mathcal{Y}^*, \mathbb{R}^{|I_l|}\right). \end{cases}$$

a) Let $(\bar{h}, v) \in L^s(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{V}_s$. Then there exists $\tilde{v} \in \mathcal{V}_s$ such that

$$\begin{cases} \tilde{v} = v \text{ on } J_0 \cup \cdots \cup J_{l-1}, \\ M_t \tilde{v}_t + N_t\left(z[\tilde{v}, z_0]_t, \tilde{v}, z[\tilde{v}, z_0]\right) = \bar{h}_t \text{ for a.a. } t \in J_l. \end{cases} \tag{3.60}$$

b) Let $(\bar{h}, \bar{v}) \in L^s(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{V}_s$ be such that

$$M_t \bar{v}_t + N_t\left(z[\bar{v}, z_0]_t, \bar{v}, z[\bar{v}, z_0]\right) = \bar{h}_t \text{ for a.a. } t \in J_l.$$

Let $(h^k, v^k) \in L^\infty(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{U}$, $k \in \mathbb{N}$, be such that $(h^k, v^k) \xrightarrow{L^s \times L^s} (\bar{h}, \bar{v})$. Then there exists $\tilde{v}^k \in \mathcal{U}$, $k \in \mathbb{N}$, such that $\tilde{v}^k \xrightarrow{L^s} \bar{v}$ and

$$\begin{cases} \tilde{v}^k = v^k \text{ on } J_0 \cup \cdots \cup J_{l-1}, \\ M_t \tilde{v}_t^k + N_t\left(z[\tilde{v}^k, z_0]_t, \tilde{v}^k, z[\tilde{v}^k, z_0]\right) = h_t^k \text{ for a.a. } t \in J_l. \end{cases} \tag{3.61}$$

Proof of Lemma 3.4.2. In the sequel we omit z_0 in the notations.

a) Let $\bar{b} \in W^{(q),s}(\Delta^\varepsilon)$. We need to find $v \in \mathcal{V}_s$ such that

$$g_i'(\bar{y})z[v] = \bar{b}_i \text{ on } \Delta_i^\varepsilon, \; i = 1, \dots, r. \tag{3.62}$$

Since

$$v = v' \text{ on } [0,t] \Longrightarrow z[v] = z[v'] \text{ on } [0,t],$$

let us construct $v^0, \dots, v^\kappa \in \mathcal{V}_s$ such that, for all l,

$$\begin{cases} v^l = v^{l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ g_i'(\bar{y})z[v^l] = \bar{b}_i \text{ on } \Delta_i^\varepsilon \cap J_l, \; i = 1, \dots, r \end{cases}$$

and $v := v^\kappa$ will satisfy (3.62).

By assumption (A4), $J_0 = [0, \tau_1 - \varepsilon_0[$ where τ_1 is the first junction time and then $\Delta_i^\varepsilon \cap J_0 = \emptyset$ for all i; we choose $v^0 := 0$.

Suppose we have $v^0, \dots v^{l-1}$ for some $l \geq 1$ and let us construct v^l. We apply Lemma 3.4.3 a) to $(\bar{b}, v^{l-1})$ with $\{t_0\} = \overline{J_{l-1}} \cap \overline{J_l}$ and we get $\tilde{b} \in W^{(q),s}([0,T])$. Since $\Delta_i^\varepsilon \cap J_l = \emptyset$ if $i \notin I_l$, it is now enough to find v^l such that

$$\begin{cases} v^l = v^{l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ g_i'(\bar{y})z[v^l] = \tilde{b}_i \text{ on } J_l, \; i \in I_l. \end{cases} \tag{3.63}$$

Suppose that $v^l = v^{l-1}$ on $J_0 \cup \dots \cup J_{l-1}$. Then $g_i'(\bar{y})z[v^l] = \tilde{b}_i$ on J_{l-1}, and it follows that

$$g_i'(\bar{y})z[v^l] = \tilde{b}_i \text{ on } J_l \tag{3.64}$$

$$\Updownarrow$$

$$\frac{d^{q_i}}{dt^{q_i}} g_i'(\bar{y})z[v^l] = \frac{d^{q_i}}{dt^{q_i}} \tilde{b}_i \text{ on } J_l. \tag{3.65}$$

And by Lemma 3.2.10, (3.65) is equivalent to

$$D_{\tilde{u}} g_i^{(q_i)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) v_t^l + D_{(\tilde{y},u,y)} g_i^{(q_i)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y})(z[v^l]_t, v^l, z[v^l]) = \tilde{b}_i^{(q_i)}(t)$$

for a.a. $t \in J_l$.

If $I_l = \emptyset$, we choose $v^l := v^{l-1}$. Otherwise, say $I_l = \{i_1 < \dots < i_p\}$ and define on J_l

$$\bar{h} := \begin{pmatrix} \tilde{b}_{i_1}^{(q_{i_1})} \\ \vdots \\ \tilde{b}_{i_p}^{(q_{i_p})} \end{pmatrix} \in L^s(J_l; \mathbb{R}^{|I_l|}).$$

Then (3.63) is equivalent to

$$\begin{cases} v^l = v^{l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ M_t v_t^l + N_t(z[v^l]_t, v^l, z[v^l]) = \bar{h}_t \text{ for a.a. } t \in J_l. \end{cases} \tag{3.66}$$

Applying Lemma 3.4.4 a) to (h, v^{l-1}), we get $\tilde{v}$ such that (3.66) holds; we choose $v^l := \tilde{v}$.

b) We follow a similar scheme to the one of the proof of a).

Let $(\bar{b},\bar{v}) \in W^{(q),2}(\Delta^\varepsilon) \times \mathcal{V}_2$ be such that

$$g_i'(\bar{y})z[\bar{v}] = \bar{b}_i \text{ on } \Delta^\varepsilon,\ i = 1,\dots,r.$$

Let $b^k \in W^{(q),\infty}(\Delta^\varepsilon)$, $k \in \mathbb{N}$, be such that $b^k \xrightarrow{W^{(q),2}} \bar{b}$. Let us construct $v^{k,0},\dots,v^{k,\kappa} \in \mathcal{U}$, $k \in \mathbb{N}$, such that for all l, $v^{k,l} \xrightarrow[k\to\infty]{L^2} \bar{v}$ and

$$\begin{cases} v^{k,l} = v^{k,l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ g_i'(\bar{y})z[v^{k,l}] = b_i^k \text{ on } \Delta_i^\varepsilon \cap J_l,\ i \in I_l. \end{cases}$$

We will conclude the proof by defining $v^k := v^{k,\kappa}$, $k \in \mathbb{N}$.

We choose for $v^{k,0}$ the truncation of $\bar{v}$, $k \in \mathbb{N}$ (see Definition 3.A.5 in Appendix 3.A.2).

Suppose we have $v^{k,0},\dots,v^{k,l-1}$, $k \in \mathbb{N}$, for some $l \geq 1$ and let us construct $v^{k,l}$, $k \in \mathbb{N}$. We apply Lemma 3.4.3 b) to $(b^k, v^{k,l-1})$ with $\{t_0\} = \overline{J_{l-1}} \cap \overline{J_l}$ and we get, for $k \in \mathbb{N}$, $\tilde{b}^k \in W^{(q),\infty}([0,T])$. In particular,

$$\tilde{b}^k \xrightarrow{W^{(q),2}} \tilde{b} \tag{3.67}$$

where $\tilde{b} := g'(\bar{y})z[\bar{v}] \in W^{(q),2}([0,T])$. And it is now enough to find $v^{k,l}$, $k \in \mathbb{N}$, such that $v^{k,l} \xrightarrow[k\to\infty]{L^2} \bar{v}$ and

$$\begin{cases} v^{k,l} = v^{k,l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ g_i'(\bar{y})z[v^{k,l}] = \tilde{b}_i^k \text{ on } J_l,\ i \in I_l. \end{cases} \tag{3.68}$$

If $I_l = \emptyset$, we choose $v^{k,l} = v^{k,l-1}$, $k \in \mathbb{N}$. Otherwise, say $I_l = \{i_1 < \dots < i_p\}$ and define on J_l

$$\bar{h} := \begin{pmatrix} \tilde{b}_{i_1}^{(q_{i_1})} \\ \vdots \\ \tilde{b}_{i_p}^{(q_{i_p})} \end{pmatrix} \in L^2(J_l;\mathbb{R}^{|I_l|}), \quad h^k := \begin{pmatrix} (\tilde{b}_{i_1}^k)^{(q_{i_1})} \\ \vdots \\ (\tilde{b}_{i_p}^k)^{(q_{i_p})} \end{pmatrix} \in L^\infty(J_l;\mathbb{R}^{|I_l|}).$$

We have

$$M_t\bar{v}_t + N_t(z[\bar{v}]_t,\bar{v},z[\bar{v}]) = \bar{h}_t \text{ for a.a } t \in J_l$$

and (3.68) is equivalent to

$$\begin{cases} v^{k,l} = v^{k,l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ M_t v_t^{k,l} + N_t(z[v^{k,l}]_t, v^{k,l}, z[v^{k,l}]) = h_t^k \text{ for a.a. } t \in J_l. \end{cases} \tag{3.69}$$

By (3.67), $h^k \xrightarrow{L^2} \bar{h}$, and by assumption, $v^{k,l-1} \xrightarrow[k\to\infty]{L^2} \bar{v}$. Applying Lemma 3.4.4 b) to $(h^k, v^{k,l-1})$, we get $\tilde{v}^k$, $k \in \mathbb{N}$, such that $\tilde{v}^k \xrightarrow{L^2} \bar{v}$ and (3.69) holds; we choose $v^{k,l} = \tilde{v}^k$, $k \in \mathbb{N}$.

$\square$

3.4.3 Necessary conditions

Recall that we are under the assumptions (A0)-(A4).

Structure of the set of lagrange multipliers

Recall that we denote by Λ the set of Lagrange multipliers associated with $(\bar{u}, \bar{y})$ (Definition 3.2.3). We consider the projection map

$$\begin{array}{rccc} \pi: & \mathcal{M} \times \mathbb{R}^{s*} \times \mathcal{P} & \longrightarrow & \mathbb{R}^{N*} \times \mathbb{R}^{s*} \\ & (\mu, \Psi, p) & \longmapsto & \left(([\mu_{i,\tau}])_{\tau,i}, \Psi \right) \end{array}$$

where $\tau \in \mathcal{T}_i$, $i = r_1 + 1, \ldots, r$. A consequence of Lemma 3.4.2 a) is the following:

Lemma 3.4.5. $\pi|_\Lambda$ *is injective.*

Proof. We will use the fact that one of the constraint, namely G_1, has a surjective derivative. For $\rho \in \prod_{i=1}^r \mathcal{M}(\Delta_i^\varepsilon)$, we define $F_\rho \in \left(W^{(q),\infty}(\Delta^\varepsilon) \right)^*$ by

$$F_\rho(\varphi) := \sum_{1 \le i \le r} \int_{\Delta_i^\varepsilon} \varphi_{i,t} \mathrm{d}\rho_{i,t} \quad \text{for all } \varphi \in W^{(q),\infty}(\Delta^\varepsilon).$$

Since by Lemma 3.2.10, $DG_1(\bar{u}, \bar{y}_0)(v, z_0) \in W^{(q),\infty}(\Delta^\varepsilon)$ for all $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$, we have

$$\begin{aligned} \langle \rho, DG_1(\bar{u}, \bar{y}_0)(v, z_0) \rangle &= \langle F_\rho, DG_1(\bar{u}, \bar{y}_0)(v, z_0) \rangle \\ &= \langle (DG_1(\bar{u}, \bar{y}_0))^* F_\rho, (v, z_0) \rangle . \end{aligned}$$

Then differentiating L_R, defined by (3.41), w.r.t. (u, y_0) we get

$$\begin{aligned} & D_{(u,y_0)} L_R(\bar{u}, \bar{y}_0, \rho, \nu, \Psi) \\ & \qquad = DJ(\bar{u}, \bar{y}_0) + DG_1(\bar{u}, \bar{y}_0)^* F_\rho + DG_2(\bar{u}, \bar{y}_0)^* \nu + DG_3(\bar{u}, \bar{y}_0)^* \Psi. \quad (3.70) \end{aligned}$$

Let $(\mu, \Psi, p), (\mu', \Psi', p') \in \Lambda$ be such that $\pi((\mu, \Psi, p)) = \pi((\mu', \Psi', p'))$. By Lemma 3.3.11, let $(\rho, \nu), (\rho', \nu')$ be such that $(\rho, \nu, \Psi), (\rho', \nu', \Psi') \in \Lambda_R$. Then $(\nu, \Psi) = (\nu', \Psi')$, and by definition of Λ_R,

$$D_{(u,y_0)} L_R(\bar{u}, \bar{y}_0, \rho, \nu, \Psi) = D_{(u,y_0)} L_R(\bar{u}, \bar{y}_0, \rho', \nu, \Psi) = 0.$$

Then by (3.70), $DG_1(\bar{u}, \bar{y}_0)^* F_\rho = DG_1(\bar{u}, \bar{y}_0)^* F_{\rho'}$. And as a consequence of Lemma3.4.2 a), $DG_1(\bar{u}, \bar{y}_0)^*$ is injective on $\left(W^{(q),\infty}(\Delta^\varepsilon) \right)^*$. Then $F_\rho = F_{\rho'}$, and by density of $W^{(q),\infty}(\Delta^\varepsilon)$ in $\prod C(\Delta_i^\varepsilon)$, we get $\rho = \rho'$. Together with $\nu = \nu'$, it implies $\mu = \mu'$ and then $(\mu, \Psi, p) = (\mu', \Psi', p')$. $\square$

As a corollary, we get a refinement of Theorem 3.3.4:

Theorem 3.4.6. *Let $(\bar{u}, \bar{y})$ be a qualified local solution of (P). Then Λ is nonempty, convex, of finite dimension and compact.*

Proof. Let $\Lambda_\pi := \pi(\Lambda)$. By Theorem 3.3.4, Λ is nonempty, convex, weakly $*$ compact and Λ_π is nonempty, convex, of finite dimension and compact (π is linear continuous and its values lie in a finite-dimensional vector space). By Lemma 3.4.5, $\pi|_\Lambda \colon \Lambda \to \Lambda_\pi$ is a bijection. We claim that its inverse

$$\begin{array}{rccc} m: & \Lambda_\pi & \longrightarrow & \Lambda \\ & (([\mu_{i,\tau}])_{\tau,i}, \Psi) & \longmapsto & (\mu, \Psi, p) \end{array}$$

is the restriction of a continuous affine map. Since $\Lambda = m(\Lambda_\pi)$, the result follows. For the claim, using the convexity of both Λ_π and Λ, the linearity of π and its injectivity when restricted to Λ, we get that m preserves convex combinations of elements from Λ_π. Thus we can extend it to an affine map on the affine subspace of $\mathbb{R}^{N*} \times \mathbb{R}^{s*}$ spanned by Λ_π. Since this subspace is of finite dimension, the extension of m is continuous. $\square$

Second-order conditions on a large critical cone

Recall that for $\lambda \in \Lambda$, $\Omega[\lambda]$ has been defined on $\mathcal{U} \times \mathbb{R}^n$ by (3.53) or (3.54).

Remark 3.4.7. The form Ω is quadratic w.r.t. (v, z_0) and affine w.r.t. λ. By Lemmas 3.2.4, 3.2.5 and 3.2.10, $\Omega[\lambda]$ can be extended continuously to $\mathcal{V}_2 \times \mathbb{R}^n$ for any $\lambda \in \Lambda$. We obtain the so-called *Hessian of Lagrangian*

$$\Omega[\lambda] \colon \mathcal{V}_2 \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

which is jointly continuous w.r.t. λ and (v, z_0).

The critical L^2 cone C_2 has been defined by (3.55)-(3.57). Let the *strict critical L^2 cone* be the set

$$C_2^S := \left\{ (v, z_0) \in C_2 \, : \, g_i'(\bar{y})z = 0 \text{ on } \Delta_i, \ i = 1, \dots, r \right\},$$

where $z = z[v, z_0] \in \mathcal{Z}_2$.

Theorem 3.4.8. *Let $(\bar{u}, \bar{y})$ be a qualified local solution of (P). Then for any $(v, z_0) \in C_2^S$, there exists $\lambda \in \Lambda$ such that*

$$\Omega[\lambda](v, z_0) \geq 0.$$

The proof is based on the following density lemma, announced in the introduction and proved in the next section:

Lemma 3.4.9. *$C_\infty^R \cap C_2^S$ is dense in C_2^S for the $L^2 \times \mathbb{R}^n$ norm.*

Proof of Theorem 3.4.8. Let $(v, z_0) \in C_2^S$. By Lemma 3.4.9, there exists a sequence $(v^k, z_0^k) \in C_\infty^R \cap C_2^S$, $k \in \mathbb{N}$, such that

$$(v^k, z_0^k) \longrightarrow (v, z_0).$$

By Lemma 3.3.13, there exists a sequence $\lambda^k \in \Lambda$, $k \in \mathbb{N}$, such that

$$\Omega[\lambda^k](v^k, z_0^k) \geq 0. \tag{3.71}$$

By Theorem 3.4.6, Λ is strongly compact; then there exists $\lambda \in \Lambda$ such that, up to a subsequence,

$$\lambda^k \longrightarrow \lambda.$$

We conclude by passing to the limit in (3.71), thanks to Remark 3.4.7. □

A density result

In this section we prove Lemma 3.4.9, using Lemma 3.4.2 b). A result similar to Lemma 3.4.9 is stated, in the framework of ODEs, as Lemma 5 in [21], but the proof given there is wrong. Indeed, the costates in the optimal control problems of steps a) and c) are actually not of bounded variations and thus the solutions are not essentially bounded. It has to be highlighted that in Lemma 3.4.2 b) we get a sequence of essentially bounded v^k.

Proof of Lemma 3.4.9. We define one more cone:

$$C_\infty^{R+} = \left\{ (v, z_0) \in C_\infty^R \cap C_2^S \, : \, \exists \delta > 0 \, : \, g_i'(\bar{y})z[v, z_0] = 0 \text{ on } \Delta_i^\delta, \ i = 1, \dots, r \right\},$$

and we show actually that C_∞^{R+} is dense in C_2^S.

To do so, we consider the following two normed vector spaces:

$$X_\infty^+ := \left\{ (v, z_0) \in \mathcal{U} \times \mathbb{R}^n \, : \, \exists \delta > 0 \, : \, g_i'(\bar{y}) z[v, z_0] = 0 \text{ on } \Delta_i^\delta, \ i = 1, \ldots, r \right\},$$
$$X_2 := \left\{ (v, z_0) \in \mathcal{V}_2 \times \mathbb{R}^n \, : \, g_i'(\bar{y}) z[v, z_0] = 0 \text{ on } \Delta_i, \ i = 1, \ldots, r \right\}.$$

Observe that C_∞^{R+} and C_2^S are defined as the same polyhedral cone by (3.56)-(3.57), respectively in X_∞^+ and X_2. In view of Lemma 1 in [38], it is then enough to show that X_∞^+ is dense in X_2.

We will need the following lemma, proved in Appendix 3.A.3:

Lemma 3.4.10. *Let $\bar{b}_i \in W^{q_i,2}(\Delta_i^\varepsilon)$ be such that*

$$\bar{b}_i = 0 \text{ on } \Delta_i.$$

Then there exists $b_i^\delta \in W^{q_i,\infty}(\Delta_i^\varepsilon)$, $\delta \in]0, \varepsilon[$, such that $b_i^\delta \xrightarrow[\delta \to 0]{W^{q_i,2}} \bar{b}_i$ and

$$b_i^\delta = 0 \text{ on } \Delta_i^\delta.$$

Going back to the proof of Lemma 3.4.9, let $(\bar{v}, \bar{z}_0) \in X_2$ and $\bar{b} := (\mathcal{A}_{2,\bar{z}_0} \bar{v})|_{\Delta^\varepsilon}$. We consider a sequence $\delta_k \searrow 0$ and for $i = 1, \ldots, r$, $b_i^k := b_i^{\delta_k} \in W^{q_i,\infty}(\Delta_i^\varepsilon)$ given by Lemma 3.4.10. Applying Lemma 3.4.2 b) to b^k, we get v^k, $k \in \mathbb{N}$. We have $(v^k, \bar{z}_0) \in X_\infty^+$ and $(v^k, \bar{z}_0) \longrightarrow (\bar{v}, \bar{z}_0)$. The proof is completed. □

3.4.4 Sufficient conditions

We still are under the assumptions (A0)-(A4).

Definition 3.4.11. A quadratic form Q over a Hilbert space X is a *Legendre form* iff it is weakly lower semi-continuous and if it satisfies the following property: if $x^k \rightharpoonup x$ weakly in X and $Q(x^k) \to Q(x)$, then $x^k \to x$ strongly in X.

Theorem 3.4.12. *Suppose that for any $(v, z_0) \in C_2$, there exists $\lambda \in \Lambda$ such that $\Omega[\lambda]$ is a Legendre form and*

$$\Omega[\lambda](v, z_0) > 0 \quad \text{if } (v, z_0) \neq 0. \tag{3.72}$$

Then $(\bar{u}, \bar{y})$ is a local solution of (P) satisfying the following quadratic growth condition: there exists $\beta > 0$ and $\alpha > 0$ such that

$$J(u, y_0) \geq J(\bar{u}, \bar{y}_0) + \frac{1}{2} \beta \left(\|u - \bar{u}\|_2 + |y_0 - \bar{y}_0| \right)^2 \tag{3.73}$$

for any trajectory (u, y) feasible for (P) and such that $\|u - \bar{u}\|_\infty + |y_0 - \bar{y}_0| \leq \alpha$.

Remark 3.4.13. Let $\lambda = (\mu, \Psi, p) \in \Lambda$. The *strengthened Legendre-Clebsch condition*

$$\exists \bar{\alpha} > 0 \, : \, D_{uu}^2 H[p](t, \bar{u}_t, \bar{y}_t) \geq \bar{\alpha} I_m \text{ for a.a. } t \in [0, T] \tag{3.74}$$

is satisfied iff $\Omega[\lambda]$ is a Legendre form (it can be proved by combining Theorem 11.6 and Theorem 3.3 in [53]).

Proof of Theorem 3.4.12. (i) Let us assume that (3.72) holds but that (3.73) does not. Then there exists a sequence of feasible trajectories (u^k, y^k) such that

$$\begin{cases} (u^k, y_0^k) \xrightarrow{L^\infty \times \mathbb{R}^n} (\bar{u}, \bar{y}_0), \ (u^k, y_0^k) \neq (\bar{u}, \bar{y}_0), \\ J(u^k, y_0^k) \leq J(\bar{u}, \bar{y}_0) + o\left(\|u^k - \bar{u}\|_2 + |y_0^k - \bar{y}_0|\right)^2. \end{cases} \tag{3.75}$$

Let $\sigma_k := \|u^k - \bar{u}\|_2 + |y_0^k - \bar{y}_0|$ and $(v^k, z_0^k) := \sigma_k^{-1}\left(u^k - \bar{u}, y_0^k - \bar{y}_0\right) \in \mathcal{U} \times \mathbb{R}^n$. There exists $(\bar{v}, \bar{z}_0) \in \mathcal{V}_2 \times \mathbb{R}^n$ such that, up to a subsequence,

$$(v^k, z_0^k) \rightharpoonup (\bar{v}, \bar{z}_0) \text{ weakly in } \mathcal{V}_2 \times \mathbb{R}^n.$$

(ii) We claim that $(\bar{v}, \bar{z}_0) \in C_2$.

Let $z^k := z[v^k, z_0^k] \in \mathcal{Y}$ and $\bar{z} := z[\bar{v}, \bar{z}_0] \in \mathcal{Z}_2$. We derive from the compact embedding $\mathcal{Z}_2 \subset C([0,T]; \mathbb{R}^n)$ that, up to a subsequence,

$$z^k \to \bar{z} \text{ in } C([0,T]; \mathbb{R}^n).$$

Moreover, it is classical (see e.g. the proof of Lemma 20 in [20]) that

$$J(u^k, y_0^k) = J(\bar{u}, \bar{y}_0) + \sigma_k DJ(\bar{u}, \bar{y}_0)(v^k, z_0^k) + o(\sigma_k), \tag{3.76}$$

$$g(y^k) = g(\bar{y}) + \sigma_k g'(\bar{y}) z^k + o(\sigma_k), \tag{3.77}$$

$$\Phi(y_0^k, y_T^k) = \Phi(\bar{y}_0, \bar{y}_T) + \sigma_k D\Phi(\bar{y}_0, \bar{y}_T)(z_0^k, z_T^k) + o(\sigma_k). \tag{3.78}$$

It follows that

$$DJ(\bar{u}, \bar{y}_0)(\bar{v}, \bar{z}_0) \leq 0, \tag{3.79}$$

$$\begin{cases} g_i'(\bar{y})\bar{z} \leq 0 \text{ on } \Delta_i & i = 1, \ldots, r_1, \\ g_i'(\bar{y})\bar{z} \leq 0 \text{ on } \Delta_i \cup \mathcal{T}_i & i = r_1 + 1, \ldots, r. \end{cases} \tag{3.80}$$

$$D\Phi(\bar{y}_0, \bar{y}_T)(\bar{z}_0, z[\bar{v}, \bar{z}_0]_T) \in T_K\left(\Phi(\bar{y}_0, \bar{y}_T)\right), \tag{3.81}$$

using (3.75) for (3.79) and the fact that $(\bar{u}, \bar{y})$, (u^k, y^k) are feasible for (3.80) and (3.81). By Lemma 3.3.1, given $\bar{\lambda} = (\bar{\mu}, \bar{\Psi}, \bar{p}) \in \Lambda$, we have

$$DJ(\bar{u}, \bar{y}_0)(\bar{v}, \bar{z}_0) + \int_{[0,T]} \mathrm{d}\bar{\mu}_t g'(\bar{y}_t) + \bar{\Psi} D\Phi(\bar{y}_0, \bar{y}_T)(\bar{z}_0, \bar{z}_T) = 0.$$

Together with Definition 3.2.3 and (3.79)-(3.81), it implies that each of the three terms is null, i.e. $(\bar{v}, \bar{z}_0) \in C_2$.

(iii) Then by (3.72) there exists $\bar{\lambda} \in \Lambda$ such that $\Omega[\bar{\lambda}]$ is a Legendre form and

$$0 \leq \Omega[\bar{\lambda}](\bar{v}, \bar{z}_0). \tag{3.82}$$

In particular, $\Omega[\bar{\lambda}]$ is weakly lower semi continuous. Then

$$\Omega[\bar{\lambda}](\bar{v}, \bar{z}_0) \leq \liminf_k \Omega[\bar{\lambda}](v^k, z_0^k) \leq \limsup_k \Omega[\bar{\lambda}](v^k, z_0^k). \tag{3.83}$$

And we claim that

$$\limsup_k \Omega[\bar{\lambda}](v^k, z_0^k) \leq 0. \tag{3.84}$$

Indeed, similarly to (3.76)-(3.78), one can show that, $\bar{\lambda}$ being a multiplier,

$$L_R(u^k, y_0^k, \bar{\lambda}) - L_R(\bar{u}, \bar{y}_0, \bar{\lambda}) = \frac{1}{2}\sigma_k^2 D^2_{(u,y_0)^2} L_R(\bar{u}, \bar{y}_0, \bar{\lambda})(v^k, z_0^k)^2 + o(\sigma_k^2). \tag{3.85}$$

Since $L_R(u^k, y_0^k, \bar{\lambda}) - L_R(\bar{u}, \bar{y}_0, \bar{\lambda}) \leq J(u^k, y_0^k) - J(\bar{u}, \bar{y}_0)$, we derive from (3.75), (3.85) and Lemma 3.3.12 that

$$\Omega[\bar{\lambda}](v^k, z_0^k) \leq o(1).$$

(iv) We derive from (3.82), (3.83) and (3.84) that

$$\Omega[\bar{\lambda}](v^k, z_0^k) \longrightarrow 0 = \Omega[\bar{\lambda}](\bar{v}, \bar{z}_0).$$

By (3.72), $(\bar{v}, \bar{z}_0) = 0$. Then $(v^k, z_0^k) \longrightarrow (\bar{v}, \bar{z}_0)$ strongly in $\mathcal{V}_2 \times \mathbb{R}^n$ by definition of a Legendre form. We get a contradiction with the fact that $\|v^k\|_2 + |z_0^k| = 1$ for all k. $\square$

In view of Theorems 3.4.8 and 3.4.12 it appears that under an extra assumption, of the type of strict complementarity on the running state constraints, we can state no-gap second-order optimality conditions. We denote by $\mathrm{ri}\,(\Lambda)$ the relative interior of Λ (see Definition 2.16 in [25]).

Corollary 3.4.14. *Let $(\bar{u}, \bar{y})$ be a qualified feasible trajectory for (P). We assume that $C_2^S = C_2$ and that for any $\lambda \in \mathrm{ri}\,(\Lambda)$, the strengthened Legendre-Clebsch condition* (3.74) *holds. Then $(\bar{u}, \bar{y})$ is a local solution of (P) satisfying the quadratic growth condition* (3.73) *iff for any $(v, z_0) \in C_2 \setminus \{0\}$, there exists $\lambda \in \Lambda$ such that*

$$\Omega[\lambda](v, z_0) > 0. \tag{3.86}$$

Proof. Suppose (3.86) holds for some $\lambda \in \Lambda$; then it holds for some $\lambda \in \mathrm{ri}\,(\Lambda)$ too and now $\Omega[\lambda]$ is a Legendre form. By Theorem 3.4.12, there is locally quadratic growth.

Conversely, suppose (3.73) holds for some $\beta > 0$ and let

$$J_\beta(u, y_0) := J(u, y_0) - \frac{1}{2}\beta \left(\|u - \bar{u}\|_2 + |y_0 - \bar{y}_0|\right)^2.$$

Then $(\bar{u}, \bar{y}_0)$ is a local solution of the following optimization problem:

$$\min_{(u,y_0)\in\mathcal{U}\times\mathbb{R}^n} J_\beta(u, y_0), \quad \text{subject to} \quad G_i(u, y_0) \in K_i, \ i = 1, 2, 3.$$

This problem has the same Lagrange multipliers as the reduced problem (write that the respective Lagrangian is stationary at $(\bar{u}, \bar{y}_0)$), the same critical cones and its Hessian of Lagrangian is

$$\Omega_\beta[\lambda](v, z_0) = \Omega[\lambda](v, z_0) - \beta \left(\|v\|_2 + |z_0|\right)^2.$$

Theorem 3.4.8 applied to this problem gives (3.86). $\square$

Remark 3.4.15. A sufficient condition (not necessary *a priori*) to have $C_2^S = C_2$ is the existence of $(\bar{\mu}, \bar{\Psi}, \bar{p}) \in \Lambda$ such that

$$\mathrm{supp}(\bar{\mu}_i) = \Delta_i, \ i = 1, \ldots, r.$$

3.5 Concluding remarks

Our main result in this paper is the statement of second-order necessary conditions on a large critical cone. This result is obtained by density, under some assumptions on the running state constraints and their contact sets. The density technique might be adapted to mixed control-state constraints.

These necessary conditions turn out to be no-gap optimality conditions if a strict complementarity condition and a strengthened Legendre-Clebsch condition hold. It has to be noted that the latter would be satisfied if we could state second-order optimality conditions involving Pontryagin multipliers, as we intend to do in a future work.

An extension of the results presented here to other classes of equations with memory, such as delay differential equations, should also be possible.

3.A Appendix

3.A.1 Functions of bounded variations

The main reference here is [5], Section 3.2. Recall that with the definition given at the beginning of Section 3.2.2, for $h \in BV\left([0,T];\mathbb{R}^{n*}\right)$ there exist $h_0, h_T \in \mathbb{R}^{n*}$ such that (3.6) holds.

Lemma 3.A.1. *Let $h \in BV\left([0,T];\mathbb{R}^{n*}\right)$. Let h^l, h^r be defined for all $t \in [0,T]$ by*

$$\begin{aligned} h^l_t &:= h_0 + \mathrm{d}h([0,t[), \\ h^r_t &:= h_0 + \mathrm{d}h([0,t]). \end{aligned}$$

Then they are both in the same equivalence class of h, h^l is left continuous, h^r is right continuous and, for all $t \in [0,T]$,

$$h^l_t = h_T - \mathrm{d}h([t,T]), \tag{3.87}$$
$$h^r_t = h_T - \mathrm{d}h(]t,T]). \tag{3.88}$$

Proof. Theorem 3.28 in [5]. □

The identification between measures and functions of bounded variations that we mention at the beginning of Section 3.2.2 relies on the following:

Lemma 3.A.2. *The linear map*

$$(c,\mu) \longmapsto \Big(h\colon t \mapsto c - \mu\left([t,T]\right)\Big)$$

is an isomorphism between $\mathbb{R}^{r*} \times \mathcal{M}\left([0,T];\mathbb{R}^{r*}\right)$ and $BV\left([0,T];\mathbb{R}^{r*}\right)$, whose inverse is

$$h \longmapsto \Big(h_T, \mathrm{d}h\Big).$$

Proof. Theorem 3.30 in [5]. □

Let us now prove Lemma 3.2.2:

Proof of Lemma 3.2.2. By (3.87), a solution in $\mathcal{P}$ of (3.11) is any $p \in L^1([0,T];\mathbb{R}^{n*})$ such that, for a.e. $t \in [0,T]$,

$$p_t = D_{y_2}\Phi[\Psi](y_0, y_T) + \int_t^T D_y H[p](s, u_s, y_s)\mathrm{d}s + \int_{[t,T]} \mathrm{d}\mu_s g'(y_s).$$

We define $\Theta\colon L^1([0,T];\mathbb{R}^{n*}) \to L^1([0,T];\mathbb{R}^{n*})$ by

$$\Theta(p)_t := D_{y_2}\Phi[\Psi](y_0, y_T) + \int_t^T D_y H[p](s, u_s, y_s)\mathrm{d}s + \int_{[t,T]} \mathrm{d}\mu_s g'(y_s)$$

for a.e. $t \in [0,T]$, and we show that Θ has a unique fixed point. Let $C > 0$ such that $\|D_y f\|_\infty, \|D^2_{y,\tau} f\|_\infty \leq C$ along (u,y).

$$\begin{aligned}
|\Theta(p_1)_t - \Theta(p_2)_t| &= \left| \int_t^T \left(D_y H[p_1](s, u_s, y_s) - D_y H[p_2](s, u_s, y_s) \right) \mathrm{d}s \right| \\
&\leq C \int_t^T \left[|p_1(s) - p_2(s)| + \int_s^T |p_1(\theta) - p_2(\theta)| \mathrm{d}\theta \right] \mathrm{d}s \\
&= C \int_t^T \left[|p_1(s) - p_2(s)| + \int_t^s |p_1(s) - p_2(s)| \mathrm{d}\theta \right] \mathrm{d}s \\
&\leq C(1+T) \int_t^T |p_1(s) - p_2(s)| \mathrm{d}s.
\end{aligned}$$

We consider the family of equivalent norms on $L^1([0,T];\mathbb{R}^{n*})$

$$\|v\|_{1,K} := \|t \mapsto e^{-K(T-t)} v(t)\|_1 \quad (K \geq 0).$$

$$\begin{aligned}
\|\Theta(p_1) - \Theta(p_2)\|_{1,K} &\leq C(1+T) \int_0^T \int_t^T e^{-K(T-t)} |p_1(s) - p_2(s)| \mathrm{d}s\mathrm{d}t \\
&= C(1+T) \int_0^T e^{-K(T-s)} |p_1(s) - p_2(s)| \left[\int_0^s e^{K(t-s)} \mathrm{d}t \right] \mathrm{d}s \\
&\leq \frac{C(1+T)}{K} \|p_1 - p_2\|_{1,K}.
\end{aligned}$$

For K big enough Θ is a contraction on $L^1([0,T];\mathbb{R}^{n*})$ for $\|\cdot\|_{1,K}$; its unique fixed point is the unique solution of (3.11). $\square$

Another useful result is the following integration by parts formula:

Lemma 3.A.3. *Let $h, k \in BV([0,T])$. Then $h^l \in L^1(\mathrm{d}k)$, $k^r \in L^1(\mathrm{d}h)$ and*

$$\int_{[0,T]} h^l \mathrm{d}k + \int_{[0,T]} k^r \mathrm{d}h = h_T k_T - h_0 k_0. \tag{3.89}$$

Proof. Let $\Delta := \{0 \leq y \leq x \leq T\}$. Since $\chi_\Delta \in L^1(\mathrm{d}h \otimes \mathrm{d}k)$, we have by Fubini's Theorem (Theorem 7.27 in [44]) and Lemma 3.A.1 that $h^l \in L^1(\mathrm{d}k)$, $k^r \in L^1(\mathrm{d}h)$ and we can compute $\mathrm{d}h \otimes \mathrm{d}k(\Delta)$ in two different ways:

$$\begin{aligned}
\mathrm{d}h \otimes \mathrm{d}k(\Delta) &= \int_{[0,T]} \int_{[y,T]} \mathrm{d}h_x \mathrm{d}k_y \\
&= \int_{[0,T]} \left(h_T - h_y^l\right) \mathrm{d}k_y \\
&= h_T \left(k_T - k_0\right) - \int_{[0,T]} h_y^l \mathrm{d}k_y, \\
\mathrm{d}h \otimes \mathrm{d}k(\Delta) &= \int_{[0,T]} \int_{[0,x]} \mathrm{d}k_y \mathrm{d}h_x \\
&= \int_{[0,T]} k_x^r \mathrm{d}h_x - k_0 \left(h_T - h_0\right).
\end{aligned}$$

□

3.A.2 The hidden use of assumption (A3)

We use assumption (A3) to prove Lemma 3.4.4 (and then Lemma 3.4.2, ...) through the following:

Lemma 3.A.4. *Recall that $M_t := D_{\tilde{u}} G^{(q)}_{I^{\varepsilon_0}(t)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) \in M_{|I_t^{\varepsilon_0}|,m}(\mathbb{R})$, $t \in [0,T]$. Then $M_t M_t^*$ is invertible and $|\left(M_t M_t^*\right)^{-1}| \leq \gamma^{-2}$ for all $t \in [0,T]$.*

Proof. For any $x \in \mathbb{R}^{|I^{\varepsilon_0}(t)|}$,

$$\langle M_t M_t^* x, x \rangle = |M_t^* x|^2 \geq \gamma^2 |x|^2.$$

Then $M_t M_t^* x = 0$ implies $x = 0$ and the invertibility follows.

Let $y \in \mathbb{R}^{|I^{\varepsilon_0}(t)|}$ and $x := \left(M_t M_t^*\right)^{-1} y$.

$$|y||x| \geq \langle y, x \rangle = \langle M_t M_t^* x, x \rangle = |M_t^* x|^2 \geq \gamma^2 |x|^2.$$

For $y \neq 0$, we have $x \neq 0$; dividing the previous inequality by $|x|$, we get

$$\gamma^2 \left| \left(M_t M_t^*\right)^{-1} y \right| \leq |y|.$$

The result follows. □

Before we prove Lemma 3.4.4, we define the truncation of an integrable function:

Definition 3.A.5. Given any $\phi \in L^s(J)$ ($s \in [1, \infty[$ and J interval), we will call *truncation of ϕ* the sequence $\phi^k \in L^\infty(J)$ defined for $k \in \mathbb{N}$ and a.a. $t \in J$ by

$$\phi_t^k := \begin{cases} \phi_t & \text{if } |\phi_t| \leq k, \\ k \dfrac{\phi_t}{|\phi_t|} & \text{otherwise.} \end{cases}$$

Observe that $\phi^k \xrightarrow[k \to \infty]{L^s} \phi$.

Proof of Lemma 3.4.4. In the sequel we omit z_0 in the notations.
(i) Let $v \in \mathcal{V}_s$. We claim that v satisfies

$$M_t v_t + N_t\left(z[v]_t, v, z[v]\right) = h_t \text{ for a.a. } t \in J_l \tag{3.90}$$

iff there exists $w \in L^s(J_l;\mathbb{R}^m)$ such that (v,w) satisfies

$$\begin{cases} M_t w_t = 0, \\ v_t = M_t^*\left(M_t M_t^*\right)^{-1}\left(h_t - N_t(z[v]_t, v, z[v])\right) + w_t, \end{cases} \quad \text{for a.a. } t \in J_l. \tag{3.91}$$

Clearly, if (v,w) satisfies (3.91), then v satisfies (3.90). Conversly, suppose that v satisfies (3.90). With Lemma 3.A.4 in mind, we define $\alpha \in L^s(J_l;\mathbb{R}^{|I_l|})$ and $w \in L^s(J_l;\mathbb{R}^m)$ by

$$\begin{aligned} \alpha &:= \left(MM^*\right)^{-1} Mv, \\ w &:= \left(I_m - M^*\left(MM^*\right)^{-1} M\right) v. \end{aligned}$$

Then

$$\begin{cases} Mw = 0, \\ v = M^*\alpha + w, \end{cases} \quad \text{on } J_l. \tag{3.92}$$

We derive from (3.90) and (3.92) that

$$M_t M_t^* \alpha_t + N_t\left(z[v]_t, v, z[v]\right) = h_t \text{ for a.a. } t \in J_l.$$

Using again Lemma 3.A.4 and (3.92), we get (3.91).

(ii) Given $(v,h,w) \in \mathcal{V}_s \times L^s(J_l;\mathbb{R}^{|I_l|}) \times L^s(J_l;\mathbb{R}^m)$, there exists a unique $\tilde{v} \in \mathcal{V}_s$ such that

$$\begin{cases} \tilde{v} = v \text{ on } J_0 \cup \cdots \cup J_{l-1} \cup J_{l+1} \cup \cdots \cup J_\kappa, \\ \tilde{v}_t = M_t^*\left(M_t M_t^*\right)^{-1}\left(h_t - N_t(z[\tilde{v}]_t, \tilde{v}, z[\tilde{v}])\right) + w_t \text{ for a.a. } t \in J_l, \end{cases} \tag{3.93}$$

Indeed, one can define a mapping from $\mathcal{V}_s$ to $\mathcal{V}_s$, using the right-hand side of (3.93). Then it can be shown, as in the proof of Lemma 3.2.2, that this mapping is a contraction for a well-suited norm, using Lemmas 3.2.4, 3.2.5 and 3.A.4. The existence and uniqueness follow. Moreover, a version of the contraction mapping theorem with parameter (see e.g. Théorème 21-5 in [31]) shows that $\tilde{v}$ depends continuously on (v,h,w).

(iii) Let us prove a): let $(\bar{h},v) \in L^s(J_l;\mathbb{R}^{|I_l|}) \times \mathcal{V}_s$ and let $w := 0$. Let $\tilde{v} \in \mathcal{V}_s$ be the unique solution of (3.93) for $(v,\bar{h},w)$. Then $\tilde{v}$ is a solution of (3.60) by *(i)*.

(iv) Let us prove b): let $(\bar{h},\bar{v}) \in L^s(J_l;\mathbb{R}^{|I_l|}) \times \mathcal{V}_s$ as in the statement and let $\bar{w}$ be given by *(i)*. Then $\bar{v}$ is the unique solution of (3.93) for $(\bar{v},\bar{h},\bar{w})$.

Let $(h^k,v^k) \in L^\infty(J_l;\mathbb{R}^{|I_l|}) \times \mathcal{U}$, $k \in \mathbb{N}$, be such that $(h^k,v^k) \xrightarrow{L^s \times L^s} (\bar{h},\bar{v})$ and let $w^k \in L^\infty(J_l;\mathbb{R}^m)$, $k \in \mathbb{N}$, be the truncation of $\bar{w}$. It is obvious from Definition 3.A.5 that

$$M_t w_t^k = 0 \text{ for a.a. } t \in J_l.$$

Let $\tilde{v}^k \in \mathcal{U}$ be the unique solution of (3.93) for (v^k,h^k,w^k), $k \in \mathbb{N}$. Then by uniqueness and continuity in *(ii)*,

$$\tilde{v}^k \xrightarrow{L^s} \bar{v}.$$

And $\tilde{v}^k$ is a solution of (3.61) by *(i)*. ☐

We finish this section with an example where assumption (A3) can be satisfied or not.

Example 3.A.6. We consider the scalar Example 3.2.11.2 with $q = 1$ and $f(t, s) = f(2t-s)$:

$$y_t = \int_0^t f(2t-s)u_s \mathrm{d}s, \quad t \in [0, T],$$

where f is a continuous function and is not a polynomial, and the trajectory $(\bar{u}, \bar{y}) = (0, 0)$. Then

$$M_t = f(t) \in M_{1,1}(\mathbb{R})$$

and (A3) is satisfied iff

$$f(t) \neq 0 \quad \forall t \in [0, T].$$

3.A.3 Approximations in $W^{q,2}$

We will prove in this section Lemmas 3.4.3 and 3.4.10. First we give the statement and the proof of a general result:

Lemma 3.A.7. *Let $\hat{x} \in W^{q,2}([0, 1])$. For $j = 0, \dots, q-1$, we denote*

$$\begin{cases} \hat{\alpha}_j := \hat{x}^{(j)}(0), \\ \hat{\beta}_j := \hat{x}^{(j)}(1), \end{cases}$$

and we consider $\alpha_j^k, \beta_j^k \in \mathbb{R}^q$, $k \in \mathbb{N}$, such that $(\alpha_j^k, \beta_j^k) \longrightarrow (\hat{\alpha}_j, \hat{\beta}_j)$. Then there exists $x^k \in W^{q,\infty}([0, 1])$, $k \in \mathbb{N}$, such that $x^k \xrightarrow{W^{q,2}} \hat{x}$ and, for $j = 0, \dots, q-1$,

$$\begin{cases} (x^k)^{(j)}(0) = \alpha_j^k, \\ (x^k)^{(j)}(1) = \beta_j^k. \end{cases} \tag{3.94}$$

Proof. Given $u \in L^2([0, 1])$, we define $x_u \in W^{q,2}([0, 1])$ by

$$x_u(t) := \int_0^t \int_0^{s_1} \cdots \int_0^{s_{q-1}} u(s_q) \mathrm{d}s_q \mathrm{d}s_{q-1} \cdots \mathrm{d}s_1, \ t \in [0, 1].$$

Then $x_u^{(q)} = u$ and, for $j = 0, \dots, q-1$,

$$x_u^{(j)}(1) = \gamma_j \iff \langle a_j, u \rangle_{L^2} = \gamma_j$$

where $a_j \in C([0, 1])$ is defined by

$$a_j(t) := \frac{(1-t)^{q-1-j}}{(q-1-j)!}, \ t \in [0, 1].$$

Indeed, a straightforward induction shows that

$$x_u^{(j)}(1) = \int_0^1 \int_0^{s_{j+1}} \cdots \int_0^{s_{q-1}} u(s_q) \mathrm{d}s_q \mathrm{d}s_{q-1} \cdots \mathrm{d}s_{j+1}.$$

Then integrations by parts give the expression of the a_j. Note that the a_j $(j = 0, \dots, q-1)$ are linearly independent in $L^2([0, 1])$. Then

$$\begin{array}{cccc} A\colon & \mathbb{R}^q & \longrightarrow & L^2([0,1]) \\ & \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{q-1} \end{pmatrix} & \longmapsto & \sum_{j=0}^{q-1} \lambda_j a_j \end{array}$$

is such that A^*A is invertible (A^* is here the adjoint operator). And

$$x_u^{(j)}(1) = \gamma_j,\ j = 0, \dots, q-1 \iff A^*u = (\gamma_0, \dots, \gamma_{q-1})^T. \tag{3.95}$$

Going back to the lemma, let $\hat{u} := \hat{x}^{(q)} \in L^2([0,1])$. Observe that

$$\hat{x}(t) = \sum_{l=0}^{q-1} \frac{\hat{\alpha}_l}{l!} t^l + x_{\hat{u}}(t),\ t \in [0,1],$$

and that $A^*\hat{u} = (\hat{\gamma}_0, \dots, \hat{\gamma}_{q-1})^T$ where

$$\hat{\gamma}_j := \hat{\beta}_j - \sum_{l=j}^{q-1} \frac{\hat{\alpha}_l}{(l-j)!},\ j = 0, \dots, q-1.$$

Then we consider, for $k \in \mathbb{N}$, the truncation (Definition 3.A.5) $\hat{u}^k \in L^\infty([0,1])$ of $\hat{u}$, and

$$\gamma_j^k := \beta_j^k - \sum_{l=j}^{q-1} \frac{\alpha_l^k}{(l-j)!},\ j = 0, \dots, q-1, \tag{3.96}$$

$$\gamma^k := (\gamma_0^k, \dots, \gamma_{q-1}^k)^T,$$

$$u^k := \hat{u}^k + A(A^*A)^{-1}\left(\gamma^k - A^*\hat{u}^k\right),$$

$$x^k(t) := \sum_{l=0}^{q-1} \frac{\alpha_l^k}{l!} t^l + x_{u^k}(t),\ t \in [0,1]. \tag{3.97}$$

It is clear that $u^k \in L^\infty([0,1])$ (by definition of A); then $x^k \in W^{q,\infty}([0,T])$. Since $A^*u^k = \gamma^k$ and in view of (3.95), (3.96) and (3.97), (3.94) is satisfied. Finally, $\gamma_j^k \longrightarrow \hat{\gamma}_j$, for $j = 1$ to $q-1$; then $\gamma^k \longrightarrow A^*\hat{u}$ and $u^k \longrightarrow \hat{u}$.

□

We can also prove the following:

Lemma 3.A.8. *Let $\hat{x} \in W^{q,2}([0,1])$ be such that $\hat{x}^{(j)}(0) = 0$ for $j = 0, \dots, q-1$. Then for $\delta > 0$ there exists $x^\delta \in W^{q,\infty}([0,1])$ such that $x^\delta \xrightarrow[\delta \to 0]{W^{q,2}} \hat{x}$ and*

$$x^\delta = 0 \textit{ on } [0, \delta].$$

Proof. We consider $u^\delta \in L^\infty([0,1])$, $\delta > 0$, such that $u^\delta = 0$ on $[0,\delta]$ and $u^\delta \xrightarrow[\delta \to 0]{L^2} \hat{u} := \hat{x}^{(q)}$. Then we define $x^\delta := x_{u^\delta}$ (see the previous proof). □

Now the proof of Lemma 3.4.10 is straightforward.

Proof of Lemma 3.4.10. We observe that $\bar{b}_i = 0$ on Δ_i implies that $\bar{b}_i^{(j)} = 0$ at the end points of Δ_i for $j = 0, \dots, q_i - 1$ (note that with the definition (3.40), if one component of Δ_i is a singleton, then $q_i = 1$). Then the conclusion follows with Lemma 3.A.8 applied on each component of $\Delta_i^\varepsilon \setminus \Delta_i$. □

Finally, we use Lemma 3.A.7 to prove Lemma 3.4.3.

Proof of Lemma 3.4.3. In the sequel we omit z_0 in the notations. We define a *connection in* $W^{q,\infty}$ between ψ_1 at t_1 and ψ_2 at t_2 as any $\psi \in W^{q,\infty}([t_1,t_2])$ such that

$$\begin{cases} \psi^{(j)}(t_1) = \psi_1^{(j)}(t_1), \\ \psi^{(j)}(t_2) = \psi_2^{(j)}(t_2), \end{cases} \quad j = 0, \dots, q-1.$$

a) We define $\tilde{b}_i$ on $[0,t_0]$ by $\tilde{b}_i := g_i'(\bar{y})z[v], \quad i = 1,\dots,r$. We need to explain how we define $\tilde{b}_i$ on $]t_0,T]$, using $\bar{b}_i$ and connections, to have $\tilde{b}_i \in W^{q_i,s}([0,T])$ and $\tilde{b}_i = \bar{b}_i$ on each component of $\Delta_i^\varepsilon \cap]t_0,T]$. The construction is slightly different whether $t_0 \in \Delta_i^\varepsilon$ or not, i.e. whether $i \in I_{t_0}^\varepsilon$ or not. Note that by definition of ε_0 and of t_0, I_t^ε is constant for t in a neighbourhood of t_0. We now distinguish the 2 cases just mentioned:

1. $i \in I_{t_0}^\varepsilon$: We denote by $[t_1,t_2]$ the connected component of Δ_i^ε such that $t_0 \in]t_1,t_2[$. We derive from (3.59) that $\tilde{b}_i = \bar{b}_i$ on $[t_1,t_0]$. Then we define $\tilde{b}_i := \bar{b}_i$ on $]t_0,t_2]$.
   If Δ_i^ε has another component in $]t_2,T]$, we denote the first one by $[t_1',t_2']$. Let ψ be a connection in $W^{q_i,\infty}$ between $\bar{b}_i$ at t_2 to $\bar{b}_i$ at t_1'. We define $\tilde{b}_i := \psi$ on $]t_2,t_1'[$, $\tilde{b}_i := \bar{b}_i$ on $[t_1',t_2']$, and so forth on $]t_2',T]$.
   If Δ_i^ε has no more component, we define $\tilde{b}_i$ on what is left as a connection in $W^{q_i,\infty}$ between $\bar{b}_i$ and $g_i'(\bar{y})z[v]$ at T.
2. $i \notin I_{t_0}^\varepsilon$: If Δ_i^ε has a component in $[t_0,T]$, we denote the first one by $[t_1,t_2]$. Note that $t_1 - t_0 \geq \varepsilon_0 - \varepsilon > 0$. We consider a connection in $W^{q_i,\infty}$ between $\tilde{b}_i$ at t_0 and $\bar{b}_i$ at t_1 and we continue as in 1.
   If Δ_i^ε has no component in $[t_0,T]$, we do as in 1.

b) For all $k \in \mathbb{N}$, we apply a) to (b^k,v^k) and we get $\tilde{b}^k$. We just need to explain how we can get, for $i = 1,\dots,r$,

$$\tilde{b}_i^k \xrightarrow[k\to\infty]{W^{q_i,2}} g_i'(\bar{y})z[\bar{v}].$$

By construction we have

$$\begin{array}{ll} \text{on } [0,t_0], & \tilde{b}_i^k = g_i'(\bar{y})z[v^k] \longrightarrow g_i'(\bar{y})z[\bar{v}], \\ \text{on } \Delta_i^\varepsilon, & \tilde{b}_i^k = b_i^k \longrightarrow \bar{b}_i = g_i'(\bar{y})z[\bar{v}]. \end{array}$$

Then it is enough to show that every connection which appears when we apply a) to (b^k,v^k), for example $\psi_i^k \in W^{q_i,\infty}([t_1,t_2])$, can be chosen in such a way that

$$\psi_i^k \longrightarrow g_i'(\bar{y})z[\bar{v}] \text{ on } [t_1,t_2].$$

This is possible by Lemma 3.A.7. □

Chapter 4

Medical application

This chapter is taken from [42]:

X. Dupuis. *Optimal control of leukemic cell population dynamics.* Submitted. Inria Research Report No. 8356, August 2013.

We are interested in optimizing the co-administration of two drugs for some acute myeloid leukemias (AML), and we are looking for in vitro protocols as a first step [1]. This issue can be formulated as an optimal control problem. The dynamics of leukemic cell populations in culture is given by age-structured partial differential equations, which can be reduced to a system of delay differential equations, and where the controls represent the action of the drugs. The objective function relies on eigenelements of the uncontrolled model and on general relative entropy, with the idea to maximize the efficiency of the protocols. The constraints take into account the toxicity of the drugs. We present in this paper the modeling aspects, as well as theoretical and numerical results on the optimal control problem that we get.

1. This work is part of the DIM LSC project ALMA on the Analysis of Acute Myeloid Leukemia. It brings together clinicians and biologists of the Inserm team 18 of UMRS 872 (Jean-Pierre Marie, Pierre Hirsh, Ruo-Ping Tang, Fanny Fava, Annabelle Ballesta, Faten Merhi) and mathematicians of the Inria teams Bang (Jean Clairambault, Annabelle Ballesta), Disco (Catherine Bonnet, José Luis Avila) and Commands (J. Frédéric Bonnans, Xavier Dupuis).

4.1 Introduction

Acute myeloid leukemias (AML) are cancers of the myeloid lineage of white blood cells. The process of blood production, called hematopoiesis, takes place in the bone marrow, with hematopoietic stem cells (HSC) at its root. HSC have the abilty to self-renew, i.e. to divide without differentiating, and to differentiate towards any lineage of blood cells by dividing into progenitors. These progenitors are committed stem cells which follow a path of differentiation, producing cells which are more and more engaged into one lineage and lose progressively their ability to self-renew. Once they are fully mature and functional, cells of each lineage are released into the bloodstream. The hematopoiesis consists in the regulation of the self-renewal and the differentiation of cell populations [76]. In AML, the differentiation is blocked at some early stage, leading to the accumulation of immature white blood cells, called blasts, of the myeloid lineage. This blockade being associated with a proliferation advantage, the blasts quickly crowd the bone marrow and are eventually released into the bloodstream.

One of the first mathematical model on hematopoiesis was proposed in 1978 by Mackey and focused on the HSC population dynamics [62]. Mackey considered two phases in his model, a resting phase and a proliferating phase, and described the dynamics of the two HSC sub-populations by a system of delay differential equations; these equations can be justified by age-structured partial differential equations. To represent the blockade of the differentiation in AML, Adimy *et al.* considered the dynamics of cell populations of several maturity stages and developped a multi-compartmental model, where each compartment represents a maturity stage and is again divided in two phases [2]. Özbay *et al.* proceeded with the stabiliy analysis of this delay differential system in [74], and Avila *et al.* refined the model in [7] by considering more than two phases per compartement and modeling the fast proliferation in AML. Stiehl and Marciniak also proposed a multi-compartmental model on leukemias [86]; they considered healthy and leukemic cell populations, but did not distinguish resting and proliferating phases and thus did not get delays.

The treatment for most of the types of AML is a challenge [83]. Clinicians of the department of hematology at Saint-Antoine hospital in Paris would be interested for some cases in co-administrating two drugs: a cytotoxic (Aracytin), which enhances cell death, and a cytostatic (AC200), which slows down proliferation. A first step is to determine how such a combination should be scheduled in in vitro experiments. To that purpose, biologists of the same hospital have sampled blood from patients with AML, sorted cancer blasts, and carried out leukemic cell cultures. The number of cells, their state in the cell cycle, and their maturity stage have then been daily measured during 5 days, without and with each of the two drugs at different constant concentrations in the culture [9].

In this paper, we idealize these experiments and consider leukemic cell cultures with varying concentration of both drugs. We are looking for in vitro protocols of drugs administration, i.e. schedules of the concentration of both drugs during the experiment, which are as efficient as possible without being too toxic. To formulate this issue as an optimal control problem, a state equation, an objective function, and constraints have to been set.

The state equation models the cell population dynamics under the action of the drugs; we consider an age-structured model with one maturity compartement, divided in one resting phase and one proliferating phase. Adimy and Crauste used such a model in [1] to represent the dependence of cell death and proliferation on growth factors. Here, the action of the cytotoxic on cell death is age-dependent, and the drug concentrations are not solutions of evolution equations but are control variables which define an in vitro protocol. Gabriel *et al.* identified the action of a drug inducing quiescence (erlotinib) with a fraction of quiescent cells in [46]. The action of the cytostatic in our model is also represented by a fraction of resting cells, which is here time-dependent, and not by a varying velocity in the proliferating phase as Hinow *et al.* in [56]. See also [13] about the modeling of the action of the drugs.

The objective function aims at minimizing the leukemic cell population at the end of the experiment, in order to maximize the efficiency of the corresponding protocol. Its definition actually requires a long time asymptotic analysis to avoid an horizon effect. This analysis relies on the specialization of the general relative entropy principle introduced by Michel *et al.* [68] to our model. Various kinds of objective functions exist in the litterature: final or maximal number of tumor cells [10], final tumor volume [59], performance index [60], or eigenvalue [14]; the use of an age-dependent weight given by eigenelements in this paper seems to be new.

The constraints come from biological bounds on the action of the drugs and from maximal cumulative doses that we impose to limit the toxicity of the protocols, as in [59]; there is no healthy population in our model on which we could set a toxicity threshold as in [10, 14]. The optimization problem that we get is equivalent, by the method of characteristics, to an optimal control problem of delay differential equations. For such a problem, optimality conditions are available in the form of Pontryagin's minimum principle [50]; it can also be reduced to an undelayed optimal control problem [48, 49], and then solved numerically by standard solvers.

The paper is organized as follows. In Section 4.2, we model the population dynamics under the action of the drugs. Section 4.3 contains the analysis of this model, including a general relative entropy principle and a long time asymptotic analysis. The optimal control problem is set in Section 4.4, and theoretical results and numerical optimal protocols are presented in Section 4.5. The precise statement of Pontryagin's minimum principle for our problem has been postponed to the appendix, together with the parameters used for the numerical resolutions.

4.2 Modeling

We present here the dynamics of leukemic cell populations in culture and under the action of the two drugs.

4.2.1 Cell populations

We consider a leukemic cell population in vitro and we distinguish two sub-populations [1, 62]: the resting cells, which are inactive (G_0 phase), and the proliferating cells, which are engaged in their cycle (G_1SG_2M phase).

Resting cells are introduced into the proliferating phase at a rate β, independently of the time spent in the resting phase. Considering that the proliferation is uncontrolled in case of AML, we do not represent any feedback from a cell population [62, 64] or a growth factor [1], and thus *β is constant* in our model.

Proliferating cells die by apoptosis at a rate γ, and if it does not die, a cell divide during mitosis, after a time 2τ spent in the phase, in two daughter cells which enter the resting phase. We consider that *the duration of the proliferating phase 2τ is the same for all cells*; this is not true biologically [2] but one can think of 2τ as an average duration [64].

We structure the proliferating population by an age variable a which represents the time spent in the proliferating phase by a cell. We denote by $R(t)$ the resting population at time t, and by $p(t, a)$ the proliferating population density with age a at time t.

4.2.2 Action of the drugs

The two drugs are a cytotoxic (Aracytin) and a cytostatic (AC220).

The cytotoxic damages the DNA of the cells during the S sub-phase of their cycle; it results in an extra death rate. To simplify further calculus and numerical issues, we consider that $u(t)$, the death rate due to the cytotoxic at time t, affects the second-half of the proliferating phase, i.e. proliferating cells with age $a \in [\tau, 2\tau]$.

The cytostatic inhibits a receptor tyrosine kynase (Flt3) of the cells in the resting phase; it results in a fraction $k(t)$ of inhibited cells among the resting cells, which can no more enter the proliferating phase. The global introduction rate to the proliferating phase at time t is then $(1 - k(t))\beta$.

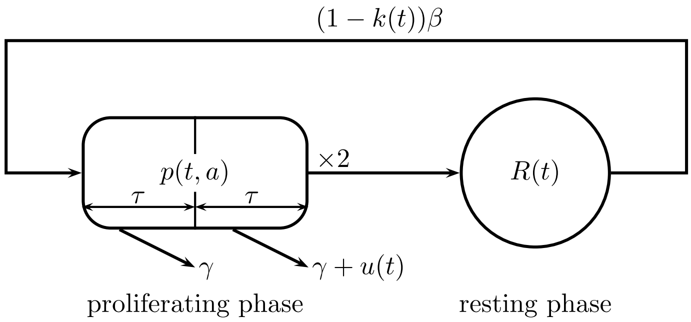

Figure 4.1: The model.

We denote by $v(t)$ the inhibition rate due to the cytostatic at time t, and by α the rate of natural dis-inhibition. We consider that the dynamics of k is given by

$$\frac{\mathrm{d}k}{\mathrm{d}t}(t) = v(t)(1 - k(t)) - \alpha k(t). \tag{4.1}$$

The action rates due to the drugs are increasing functions of their concentration in the cell culture, the latter being chosen during in vitro experiments. Thus we consider that we control directly the action rates u and v.

4.2.3 The age-structured model

The dynamics of the cell populations is given by the following partially age-structured system:

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = -(1 - k(t))\beta R(t) + 2p(t, 2\tau) \tag{4.2}$$

$$\frac{\partial p}{\partial t}(t, a) + \frac{\partial p}{\partial a}(t, a) = -(\gamma + \chi_{(\tau,2\tau)}(a)u(t))p(t, a) \qquad 0 < a < 2\tau \tag{4.3}$$

$$p(t, 0) = (1 - k(t))\beta R(t) \tag{4.4}$$

The equation (4.2) is a balance equation for the resting phase between the outward and inward flow; the transport equation (4.3) describes the evolution of the age cohorts of proliferating cells, since they are aging with velocity 1; the boundary condition (4.4) gives the inward flow to the proliferating phase.

4.2.4 A controlled version of Mackey's model

We denote by P and P_2 the total proliferating population and sub-population in the second-half of the phase, respectively:

$$P(t) := \int_0^{2\tau} p(t,a)\mathrm{d}a, \quad P_2(t) := \int_\tau^{2\tau} p(t,a)\mathrm{d}a.$$

Formally, and this could be justified with the results of Section 4.3, if we differentiate P, P_2 and use the method of characteristics as in [1], we derive from (4.1)-(4.4) the following system of delay differential equations:

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = -(1-k(t))\beta R(t) + 2(1-k(t-2\tau))\beta R(t-2\tau)e^{-\left(\gamma 2\tau + \int_{t-\tau}^t u(s)\mathrm{d}s\right)} \tag{4.5}$$

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t}(t) &= -(\gamma P(t) + u(t)P_2(t)) + (1-k(t))\beta R(t) \\ &\quad - (1-k(t-2\tau))\beta R(t-2\tau)e^{-\left(\gamma 2\tau + \int_{t-\tau}^t u(s)\mathrm{d}s\right)}\end{aligned} \tag{4.6}$$

$$\begin{aligned}\frac{\mathrm{d}P_2}{\mathrm{d}t}(t) &= -(\gamma + u(t))P_2(t) + (1-k(t-\tau))\beta R(t-\tau)e^{-\gamma\tau} \\ &\quad - (1-k(t-2\tau))\beta R(t-2\tau)e^{-\left(\gamma 2\tau + \int_{t-\tau}^t u(s)\mathrm{d}s\right)}\end{aligned} \tag{4.7}$$

$$\frac{\mathrm{d}k}{\mathrm{d}t}(t) = v(t)(1-k(t)) - \alpha k(t) \tag{4.8}$$

Unsurprisingly, we get a controlled version of Mackey's 1978 model [62]. The original model is a system of two differential equations with one discrete delay, and a nonlinearity in β; it is one of the first mathematical model of the dynamics of hematopoietic stem cells (HSC), which are at the root of the hematopoiesis, the process of blood production. We have in (4.5)-(4.8) two control variables, u and v, and two extra state variables, P_2 and k, because of the controls.

As we will explain in Section 4.4.1, the age-structure in the proliferating population actually matters, and thus it is of interest to analyse the age-structured model (4.2)-(4.4).

4.3 Analysis of the age-structured model

4.3.1 Existence of solutions

Given $(\beta,\gamma) \in L^\infty_{loc}(0,\infty) \times L^\infty_{loc}((0,\infty)\times(0,2\tau))$ and $(R_0,p_0) \in \mathbb{R} \times L^\infty(0,2\tau)$, we consider the system

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = -\beta(t)R(t) + 2p(t,2\tau) \qquad 0<t \tag{4.9}$$

$$\frac{\partial p}{\partial t}(t,a) + \frac{\partial p}{\partial a}(t,a) = -\gamma(t,a)p(t,a) \qquad 0<t,\ 0<a<2\tau \tag{4.10}$$

$$p(t,0) = \beta(t)R(t) \qquad 0<t \tag{4.11}$$

with the initial condition

$$R(0) = R_0, \qquad p(0,\cdot) = p_0. \tag{4.12}$$

We follow [43] for the notion of solution.

Definition 4.3.1. We say that (4.10) holds along the characteristics a.e. if and only if there holds, for a.a. $(t,a) \in (0,\infty) \times (0,2\tau)$,

$$p(s,a+s) = p(0,a) - \int_0^s (\gamma p)(\theta, a+\theta)\mathrm{d}\theta \qquad \text{for a.a. } s \in (0, 2\tau - a), \tag{4.13}$$

$$p(t+s,s) = p(t,0) - \int_0^s (\gamma p)(t+\theta, \theta)\mathrm{d}\theta \qquad \text{for a.a. } s \in (0, 2\tau). \tag{4.14}$$

Lemma 4.3.2. *If $p \in L^\infty_{loc}((0,\infty) \times (0,2\tau))$ is such that (4.10) holds along the characteristics a.e., then p is Lipschitz along the characteristics $\{t - a = c\}$ for a.a. c, $t \mapsto \int_0^{2\tau} p(t,a)\mathrm{d}a$ is locally Lipschitz and there holds a.e.*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\tau} p(t,a)\mathrm{d}a = p(t,0) - p(t,2\tau) - \int_0^{2\tau} (\gamma p)(t,a)\mathrm{d}a.$$

Proof. The first assertion follows from (4.13)-(4.14). For the last two assertions, it is enough to compute $\int_0^{2\tau} p(t,a)\mathrm{d}a$ using the same relations. $\square$

Definition 4.3.3. A solution of (4.9)-(4.12) is any

$$(R,p) \in W^{1,\infty}_{loc}(0,\infty) \times L^\infty_{loc}((0,\infty) \times (0,2\tau))$$

such that (4.9) holds a.e., (4.10) holds along the characteristics a.e., (4.11) holds a.e., and (4.12) holds.

Lemma 4.3.4. *Given any (β,γ) and (R_0,p_0), there exists a unique solution (R,p) of (4.9)-(4.12). If (β,γ) and (R_0,p_0) are non-negative, then (R,p) is non-negative. Moreover, defining*

$$\Gamma\colon (t,a) \mapsto \begin{cases} \int_0^t \gamma(s, a-t+s)\mathrm{d}s & \text{if } 0 < t < a < 2\tau, \\ \int_0^a \gamma(t-a+s, s)\mathrm{d}s & \text{if } 0 < a < 2\tau,\ a < t, \end{cases}$$

if β, Γ, p_0 are locally Lipschitz and if $p_0(0) = \beta(0)R_0$, then p is locally Lipschitz and $R \in W^{2,\infty}_{loc}$.

Proof. For a.a. $c \in (-2\tau, 0)$, p is determined on $\{t - a = c\}$ by

$$p(t,a) = p_0(a-t)e^{-\int_0^t \gamma(s,a-t+s)\mathrm{d}s}.$$

Then (4.9) becomes a linear ODE on $(0,2\tau)$, from which we get R, and then p on $\{t-a = c\}$ for a.a. $c \in (0,2\tau)$, and so on. The sign of (R,p) follows.

Observe that a.e. on $\{t - a > 0\}$,

$$p(t,a) = \beta(t-a)R(t-a)e^{-\int_0^a \gamma(t-a+s,s)\mathrm{d}s}.$$

The continuity of p on $\{t - a = 0\}$ is equivalent to $p_0(0) = \beta(0)R_0$. $\square$

4.3.2 General relative entropy

We introduce the dual system associated with (4.9)-(4.11)

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t}(t) = \beta(t)\Psi(t) - \beta(t)\phi(t,0) \qquad 0 < t \tag{4.15}$$

$$\frac{\partial\phi}{\partial t}(t,a) + \frac{\partial\phi}{\partial a}(t,a) = \gamma(t,a)\phi(t,a) \qquad 0 < t,\ 0 < a < 2\tau \tag{4.16}$$

$$\phi(t,2\tau) = 2\Psi(t) \qquad 0 < t \tag{4.17}$$

Solutions of (4.15)-(4.17) are defined as for the primal system, see Definition 4.3.3.

Adapting [68] to our model, we get a general relative entropy principle. Let (β, γ) be fixed. Given a solution (R, p) and a positive solution $(\hat{R}, \hat{p})$ of (4.9)-(4.11), a positive solution (Ψ, ϕ) of (4.15)-(4.17), and $H \in L^\infty_{loc}(\mathbb{R})$, we define $\mathcal{H}$ by

$$\mathcal{H}(t) := \Psi(t)\hat{R}(t)H\left(\frac{R(t)}{\hat{R}(t)}\right) + \int_0^{2\tau} \phi(t,a)\hat{p}(t,a)H\left(\frac{p(t,a)}{\hat{p}(t,a)}\right) \mathrm{d}a.$$

Theorem 4.3.5 (General Relative Entropy)**.** *Let H be locally Lipschitz and differentiable everywhere. Then $\mathcal{H}$ is locally Lipschitz and there holds a.e.*

$$\begin{aligned}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}(t) = \phi(t,2\tau)\hat{p}(t,2\tau)\Bigg[H\left(\frac{p(t,0)}{\hat{p}(t,0)}\right) - H\left(\frac{p(t,2\tau)}{\hat{p}(t,2\tau)}\right) \\ +H'\left(\frac{p(t,0)}{\hat{p}(t,0)}\right)\left(\frac{p(t,2\tau)}{\hat{p}(t,2\tau)} - \frac{p(t,0)}{\hat{p}(t,0)}\right)\Bigg]. \end{aligned} \tag{4.18}$$

Corollary 4.3.6. *Let H be convex, possibly non differentiable. Then $\mathcal{H}$ is non-increasing.*

Proof of Corollary 4.3.6. Let H be convex. Then H is locally Lipschitz and has left and right derivatives everywhere. Then, as in the proof of Theorem 4.3.5, $\mathcal{H}$ is locally Lipschitz and (4.18) holds a.e. if we replace the derivative of $\mathcal{H}$ by its right derivative and the derivative of H by its left or right derivative, depending on the sign of the right derivative of $\frac{p(t,0)}{\hat{p}(t,0)}$. Observe now that this right-hand side of (4.18) is non-positive if H is convex. □

Proof of Theorem 4.3.5. Observe that, along the characteristics a.e., there holds

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)\phi(t,a)\hat{p}(t,a) = 0, \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)\frac{p(t,a)}{\hat{p}(t,a)} = 0,$$

and then

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\right)\phi(t,a)\hat{p}(t,a)H\left(\frac{p(t,a)}{\hat{p}(t,a)}\right) = 0.$$

By Lemma 4.3.2, the second term of $\mathcal{H}$ is locally Lipschitz, with derivative a.e.

$$\phi(t,0)\hat{p}(t,0)H\left(\frac{p(t,0)}{\hat{p}(t,0)}\right) - \phi(t,2\tau)\hat{p}(t,2\tau)H\left(\frac{p(t,2\tau)}{\hat{p}(t,2\tau)}\right).$$

The first term of $\mathcal{H}$ is obviously locally Lipschitz, and a.e.

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Psi(t)\hat{R}(t)\right)H\left(\frac{R(t)}{\hat{R}(t)}\right) &= \big(-\beta(t)\phi(t,0)\hat{R}(t) + \Psi(t)2\hat{p}(t,2\tau)\big)H\left(\frac{R(t)}{\hat{R}(t)}\right) \\ &= \big(-\phi(t,0)\hat{p}(t,0) + \phi(t,2\tau)\hat{p}(t,2\tau)\big)H\left(\frac{p(t,0)}{\hat{p}(t,0)}\right),\end{aligned}$$

and

$$\begin{aligned}\Psi(t)\hat{R}(t)\frac{\mathrm{d}}{\mathrm{d}t}H\left(\frac{R(t)}{\hat{R}(t)}\right) &= \Psi(t)\hat{R}(t)H'\left(\frac{R(t)}{\hat{R}(t)}\right)\frac{1}{\hat{R}(t)}\left(\frac{\mathrm{d}R}{\mathrm{d}t}(t) - \frac{R(t)}{\hat{R}(t)}\frac{\mathrm{d}\hat{R}}{\mathrm{d}t}(t)\right) \\ &= \Psi(t)H'\left(\frac{p(t,0)}{\hat{p}(t,0)}\right)\left(2p(t,2\tau) - \frac{p(t,0)}{\hat{p}(t,0)}2\hat{p}(t,2\tau)\right) \\ &= \phi(t,2\tau)\hat{p}(t,2\tau)H'\left(\frac{p(t,0)}{\hat{p}(t,0)}\right)\left(\frac{p(t,2\tau)}{\hat{p}(t,2\tau)} - \frac{p(t,0)}{\hat{p}(t,0)}\right).\end{aligned}$$

□

4.3.3 Eigenelements

Let $\beta > 0$ and $\gamma \geq 0$ be constant. Looking for particular solutions of (4.9)-(4.11) and (4.15)-(4.17) of the form

$$R\colon t \mapsto \bar{R}e^{\lambda t} \qquad \Psi\colon t \mapsto \bar{\Psi}e^{-\lambda t}$$
$$p\colon (t,a) \mapsto \bar{p}(a)e^{\lambda t} \qquad \phi\colon (t,a) \mapsto \bar{\phi}(a)e^{-\lambda t}$$

with $\lambda \in \mathbb{R}$ and $\bar{p}$, $\bar{\phi}$ differentiable, we get the following eigenvalue problem:

$$(\lambda+\beta)\bar{R} = 2\bar{p}(2\tau) \qquad (\lambda+\beta)\bar{\Psi} = \beta\bar{\phi}(0) \tag{4.19}$$
$$\frac{\mathrm{d}\bar{p}}{\mathrm{d}a}(a) = -(\lambda+\gamma)\bar{p}(a) \qquad \frac{\mathrm{d}\bar{\phi}}{\mathrm{d}a}(a) = (\lambda+\gamma)\bar{\phi}(a) \tag{4.20}$$
$$\bar{p}(0) = \beta\bar{R} \qquad \bar{\phi}(2\tau) = 2\bar{\Psi} \tag{4.21}$$

Equations (4.20)-(4.21) give

$$\bar{p}(a) = \beta\bar{R}e^{-(\lambda+\gamma)a}, \qquad \bar{\phi}(a) = 2\bar{\Psi}e^{(\lambda+\gamma)(a-2\tau)}. \tag{4.22}$$

If $\bar{R}, \bar{\Psi} \neq 0$, (4.19) is then equivalent to

$$\lambda + \beta = 2\beta e^{-(\lambda+\gamma)2\tau}. \tag{4.23}$$

Theorem 4.3.7 (First eigenelements)**.** *There exists a unique solution* $(\lambda, \bar{R}, \bar{p}, \bar{\Psi}, \bar{\phi})$ *of* (4.19)-(4.21) *such that*

$$\bar{R} > 0, \quad \bar{p} > 0, \quad \bar{R} + \int_0^{2\tau} \bar{p}(a)\mathrm{d}a = 1,$$
$$\bar{\Psi} > 0, \quad \bar{\phi} > 0, \quad \bar{\Psi}\bar{R} + \int_0^{2\tau} \bar{\phi}(a)\bar{p}(a)\mathrm{d}a = 1.$$

Proof. It is enough to observe that (4.23) has a unique real solution. □

4.3.4 Long time asymptotic

without the action of the drugs

We consider here that there is no action of the drugs for $t > 0$, i.e. that $\beta > 0$ and $\gamma \geq 0$ are constant. The first eigenelements $(\lambda, \bar{R}, \bar{p}, \bar{\Psi}, \bar{\phi})$ are given by Theorem 4.3.7.

Theorem 4.3.8. *Let* (R_0, p_0) *be an initial condition,* $C > 0$ *be such that*

$$|R_0| \leq C\bar{R}, \quad |p_0(\cdot)| \leq C\bar{p}(\cdot),$$

and (R, p) *be the solution of* (4.9)-(4.12)*. Then, for all* $t > 0$*,*

$$|R(t)| \leq C\bar{R}e^{\lambda t}, \quad |p(t,\cdot)| \leq C\bar{p}(\cdot)e^{\lambda t}, \tag{4.24}$$
$$\left(\bar{\Psi}R(t) + \int_0^{2\tau} \bar{\phi}(a)p(t,a)\mathrm{d}a\right)e^{-\lambda t} = \bar{\Psi}R_0 + \int_0^{2\tau} \bar{\phi}(a)p_0(a)\mathrm{d}a := \rho, \tag{4.25}$$
$$\left(\bar{\Psi}|R(t)| + \int_0^{2\tau} \bar{\phi}(a)|p(t,a)|\mathrm{d}a\right)e^{-\lambda t} \leq \bar{\Psi}|R_0| + \int_0^{2\tau} \bar{\phi}(a)|p_0(a)|\mathrm{d}a, \tag{4.26}$$
$$\lim_{t\to\infty}\left(\bar{\Psi}|R(t)e^{-\lambda t} - \rho\bar{R}| + \int_0^{2\tau} \bar{\phi}(a)|p(t,a)e^{-\lambda t} - \rho\bar{p}(a)|\mathrm{d}a\right) = 0. \tag{4.27}$$

Remark 4.3.9. Theorem 4.3.8 gives an interpretation of the first eigenelements: for the L^1 topology,

$$(R(t), p(t,\cdot)) \sim \rho\,(\bar{R}e^{\lambda t}, \bar{p}(\cdot)e^{\lambda t})$$

as $t \to \infty$, with ρ given by (4.25). Then the first eigenvalue λ is the *Malthus parameter* of the model, which gives the overall exponential growth or decay of the population. We derive from its definition (4.23) that λ has the same sign as $2e^{-\gamma 2\tau} - 1$, which is the *proliferating phase balance*. Asymptotically, any solution becomes proportional to a particular solution with age profile given by the first primal eigenvector $(\bar{R}, \bar{p})$ and rate of time evolution λ. The coefficient of proportionality ρ is determined initially with the first dual eigenvector $(\bar{\Psi}, \bar{\phi})$.

Proof. We follow the same scheme as in [68, 77]. We apply the general relative entropy principle to (R,p), $(\bar{R}e^{\lambda t}, \bar{p}e^{\lambda t})$, $(\bar{\Psi}e^{-\lambda t}, \bar{\phi}e^{-\lambda t})$, and to the following convex functions:

- $H(h) := (h \pm C)^2_{\mp}$ for (4.24). The two corresponding entropies $\mathcal{H}$ are non-increasing by Corollary 4.3.6, non-negative, and initially null; then they are null everywhere.
- $H(h) := h$ for (4.25); $\mathcal{H}$ is constant by Theorem 4.3.5.
- $H(h) := |h|$ for (4.26); $\mathcal{H}$ is non-increasing by Corollary 4.3.6.
- $H(h) := |h - \rho|$ for (4.27); $\mathcal{H}$ is non-increasing by Corollary 4.3.6 and non-negative. Then it has a limit L, i.e.

$$\left(\bar{\Psi}|R(t)e^{-\lambda t} - \rho\bar{R}| + \int_0^{2\tau} \bar{\phi}(a)|p(t,a)e^{-\lambda t} - \rho\bar{p}(a)|\mathrm{d}a\right) \to L \tag{4.28}$$

as $t \to \infty$. It remains to prove that $L = 0$; we do that in several steps.

1. Let p_0^k, $k \in \mathbb{N}$, be Lipschitz, with $p_0^k(0) = \beta R_0$ and such that, as $k \to \infty$,

$$\varepsilon_k := \int_0^{2\tau} \bar{\phi}(a)\left|p_0^k(a) - p_0(a)\right|\mathrm{d}a \to 0.$$

Let (R^k, p^k) be the solution of (4.9)-(4.11) with initial condition (R_0, p_0^k), ρ_k be given by (4.25), and L_k be given by (4.28). Then $|\rho_k - \rho| \le \varepsilon_k$, and applying (4.26) to the solution $(R^k - R, p^k - p)$ of (4.9)-(4.11), we get

$$\left(\bar{\Psi}|R^k(t) - R(t)| + \int_0^{2\tau} \bar{\phi}(a)|p^k(t,a) - p(t,a)|\mathrm{d}a\right)e^{-\lambda t} \le \varepsilon_k$$

for all t. Then $L \le L_k + 2\varepsilon_k$, and it is enough to show that $L = 0$ for an initial condition (R_0, p_0) with p_0 Lipschitz and $p_0(0) = \beta R_0$, as we assume in the sequel of the proof.

2. Since β and γ are constant, $(R,p) \in W^{2,\infty}_{loc} \times W^{1,\infty}_{loc}$ by Lemma 4.3.4. We observe moreover that $(\frac{\mathrm{d}R}{\mathrm{d}t}, \frac{\partial p}{\partial t})$ is a solution of (4.9)-(4.11). Then by (4.24), there exists $C' > 0$ such that for all t,

$$\left|\frac{\mathrm{d}R}{\mathrm{d}t}(t)\right| \le C'\bar{R}e^{\lambda t}, \quad \left|\frac{\partial p}{\partial t}(t,\cdot)\right| \le C'\bar{p}(\cdot)e^{\lambda t}. \tag{4.29}$$

It follows by (4.10) that for all t,

$$\left|\frac{\partial p}{\partial a}(t,\cdot)\right| \le (\gamma C + C')\bar{p}(\cdot)e^{\lambda t}. \tag{4.30}$$

3. We apply again Theorem 4.3.5 to (R,p), $(\bar{R}e^{\lambda t},\bar{p}e^{\lambda t})$, $(\bar{\Psi}e^{-\lambda t},\bar{\phi}e^{-\lambda t})$, and to $H(h) := (h-1)^2$. Using that

$$H(h_1) - H(h_2) + H'(h_1)(h_2 - h_1) = -(h_1 - h_2)^2,$$

we get for the corresponding $\mathcal{H}$

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}(t) = -\bar{\phi}(2\tau)\bar{p}(2\tau)\left(\frac{p(t,2\tau)e^{-\lambda t}}{\bar{p}(2\tau)} - \frac{p(t,0)e^{-\lambda t}}{\bar{p}(0)}\right)^2.$$

Then $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}$ is globally Lipschitz by (4.24) and (4.29), and non-positive. Since $\mathcal{H}$ is bounded below by 0, it has a limit and then $\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}(t) \to 0$, i.e.

$$\frac{p(t,2\tau)e^{-\lambda t}}{\bar{p}(2\tau)} - \frac{p(t,0)e^{-\lambda t}}{\bar{p}(0)} \to 0 \tag{4.31}$$

as $t \to \infty$.

4. We define $(Q^k, n^k) \in C([0,1]) \times C([0,1]\times[0,2\tau])$, $k \in \mathbb{N}$, by

$$Q^k(t) := R(t+k)e^{-\lambda(t+k)}, \quad n^k(t,a) := p(t+k,a)e^{-\lambda(t+k)}.$$

We derive from (4.24),(4.29)-(4.30) and Arzelà-Ascoli theorem that there exists $(\bar{Q},\bar{n})$ such that, up to a subsequence, $(Q^k,n^k) \to (\bar{Q},\bar{n})$ uniformly. Then for all $t \in [0,1]$,

$$\bar{\Psi}\bar{Q}(t) + \int_0^{2\tau} \bar{\phi}(a)\bar{n}(t,a)\mathrm{d}a = \rho, \tag{4.32}$$

$$\bar{\Psi}|\bar{Q}(t) - \rho\bar{R}| + \int_0^{2\tau} \bar{\phi}(a)|\bar{n}(t,a) - \rho\bar{p}(a)|\mathrm{d}a = L, \tag{4.33}$$

$$\frac{\bar{n}(t,2\tau)}{\bar{p}(2\tau)} - \frac{\bar{n}(t,0)}{\bar{p}(0)} = 0 \tag{4.34}$$

by (4.25), (4.28), and (4.31), respectively. Moreover $(\bar{Q},\bar{n})$ is solution, in the sense of Definition 4.3.3, of

$$\frac{\mathrm{d}Q}{\mathrm{d}t}(t) = -(\lambda+\beta)Q(t) + 2n(t,2\tau) \qquad 0<t<1 \tag{4.35}$$

$$\frac{\partial n}{\partial t}(t,a) + \frac{\partial n}{\partial a}(t,a) = -(\lambda+\gamma)n(t,a) \qquad 0<t<1,\ 0<a<2\tau \tag{4.36}$$

$$n(t,0) = \beta Q(t) \qquad 0<t<1 \tag{4.37}$$

Injecting (4.34) and (4.37) into (4.35), we get

$$\frac{\mathrm{d}\bar{Q}}{\mathrm{d}t}(t) = \big[-(\lambda+\beta)\bar{p}(0) + 2\beta\bar{p}(2\tau)\big]\frac{\bar{Q}(t)}{\bar{p}(0)} = 0$$

by definition (4.22)-(4.23) of the eigenelements. Then $\bar{Q}$ is constant and

$$\begin{aligned}\bar{n}(0,a) &= \lim_k p(k,a)e^{-\lambda k}\\ &= \lim_k \beta R(k-a)e^{-\lambda(k-a)}e^{-(\lambda+\gamma)a}\\ &= \beta\bar{Q}e^{-(\lambda+\gamma)a}.\end{aligned}$$

Solving (4.36) along the characteristics, it comes that

$$\bar{n}(t,a) = \beta \bar{Q} e^{-(\lambda+\gamma)a}$$

for all $(t,a) \in [0,1] \times [0,2\tau]$; in particular, $\bar{n}$ does not depend on t. Observe that $(\bar{Q}, \bar{n})$ is proportional to $(\bar{R}, \bar{p})$; by (4.32), $(\bar{Q}, \bar{n}) = \rho(\bar{R}, \bar{p})$, and then by (4.33), $L = 0$ as was to prove.

□

with the action of the drugs

We consider now that the drugs are not administrated for $t > 0$ but that they have a residual action. Namely, if at $t = 0$ there is a fraction $k_0 \geq 0$ of inhibited cells among the resting cells, then by (4.1), for $t > 0$,

$$\beta(t) = \left(1 - k_0 e^{-\alpha t}\right) \beta$$

with $\alpha > 0$, and $\gamma \geq 0$ is constant. We continue to use the first eigenelements given by Theorem 4.3.7, i.e. for β constant too.

Lemma 4.3.10. *Assume that the first eigenvalue λ is non-negative. Let (R_0, p_0) be a non-negative initial condition, (R, p) be the solution of* (4.9)-(4.12)*, and I be defined by*

$$I(t) := \left(\bar{\Psi} R(t) + \int_0^{2\tau} \bar{\phi}(a) p(t,a) \mathrm{d}a\right) e^{-\lambda t}.$$

Then I is locally Lipschitz, non-increasing, and for all $t > 0$,

$$e^{-\lambda \frac{k_0}{\alpha}\left(1 - e^{-\alpha t}\right)} I(0) \leq I(t) \leq I(0).$$

In particular, $I(t)$ has a limit, say $I_\infty \in [e^{-\lambda \frac{k_0}{\alpha}} I(0), I(0)]$, as $t \to \infty$.

Remark 4.3.11.

1. If $\lambda \leq 0$, then we can show the reverse inequality:

$$I(0) \leq I(t) \leq e^{-\lambda \frac{k_0}{\alpha}\left(1 - e^{-\alpha t}\right)} I(0).$$

2. If $k_0 = 0$, then β is constant and we recover result (4.25) of Theorem 4.3.8: I is constantly equal to ρ.
3. The first eigenvalue λ is still the Malthus parameter of the model, in the sense that

$$\left(R(t)e^{-\lambda' t}, p(t,\cdot)e^{-\lambda' t}\right) \to \begin{cases} 0 & \text{if } \lambda' > \lambda \\ \infty & \text{if } \lambda' < \lambda \end{cases}$$

as $t \to \infty$.

4. If there exists ρ' such that, in the sense of (4.27),

$$(R(t), p(t,\cdot)) \sim \rho' \left(\bar{R}e^{\lambda t}, \bar{p}(\cdot)e^{\lambda t}\right)$$

as $t \to \infty$, then $\rho' = I_\infty$.

Proof. We still have, along the characteristics a.e.,

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}\right)\bar{\phi}(a)e^{-\lambda t}p(t,a)=0.$$

Then as in the proof of Theorem 4.3.5, I is locally Lipschitz. And there holds a.e.,

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t}(t)&=\bar{\Psi}\left(-(\lambda+\beta(t))R(t)+2p(t,2\tau)\right)e^{-\lambda t}\\&\quad+\left(\bar{\phi}(0)p(t,0)-\bar{\phi}(2\tau)p(t,2\tau)\right)e^{-\lambda t}\\&=-\lambda k_0e^{-\alpha t}\bar{\Psi}R(t)e^{-\lambda t}\end{aligned}$$

Since $\lambda,k_0,R,p\geq 0$ (see Lemma 4.3.4), we get a.e.

$$-\lambda k_0 e^{-\alpha t}I(t)\leq\frac{\mathrm{d}I}{\mathrm{d}t}(t)\leq 0.$$

The result follows. □

4.4 The optimal control problem

We fix a time horizon $T>0$ and we consider leukemic cell cultures with varying concentrations of both drugs on $[0,T]$. As explained in Section 4.2.2, we consider that, in our in vitro model (4.1)-(4.4), we control directly the death rate u due to the cytotoxic and the inhibition rate v due to the cytostatic. Thus we call *protocol of drugs administration* any $(u,v)\in L^\infty(0,T;\mathbb{R}^2)$ satisfying the following biological bounds:

$$\begin{cases}0\leq u(t)\leq\bar{u}\\0\leq v(t)\leq\bar{v}\end{cases}\quad\text{for a.a. } t\in(0,T).\tag{4.38}$$

Note that by Lemma 4.3.4, given any protocol (u,v), there exists a unique associated state, i.e. (R,p,k) such that (4.1)-(4.4) hold.

We are looking for protocols of drugs administration which are as much efficient as possible, and not too toxic. The notion of efficiency will be handled by the objective function (Section 4.4.1), and the one of toxicity by the constraints (Section 4.4.2), in our optimal control problem (Section 4.4.3).

4.4.1 Horizon effect and age-weighted population

Since we consider only leukemic cells, an efficient protocol has to aim at the extinction of the total population. Nevertheless, if we try to minimize the total population, i.e. if we consider the problem

$$\min_{(u,v,R,p,k)}\left(R(T)+\int_0^{2\tau}p(T,a)\mathrm{d}a\right)\quad\text{subject to}\quad(4.1)\text{-}(4.4),(4.38),\tag{4.39}$$

then we observe a *horizon effect*: it is always optimal to give no cytostatic v at the end of the experiment, whatever the parameters are. It can be seen numerically and proved theoretically, and it is easily understandable: the resting cells which are introduced into the proliferating phase at time $t\in(T-2\tau,T)$ will not divide before T, but might die, which is not the case if they stay in the resting phase; it is therefore optimal to have a high global introduction rate, i.e. a low fraction of inhibited cells k, at the end. We end up at time T with a filled proliferating phase, which flows into the resting phase after T. If for example the death rate in the proliferating phase is so low that its balance is positive, i.e. that it globally produces cells after division, then the total population for this optimal protocol becomes much larger than for other protocols (see Figure 4.2), which is not satisfying.

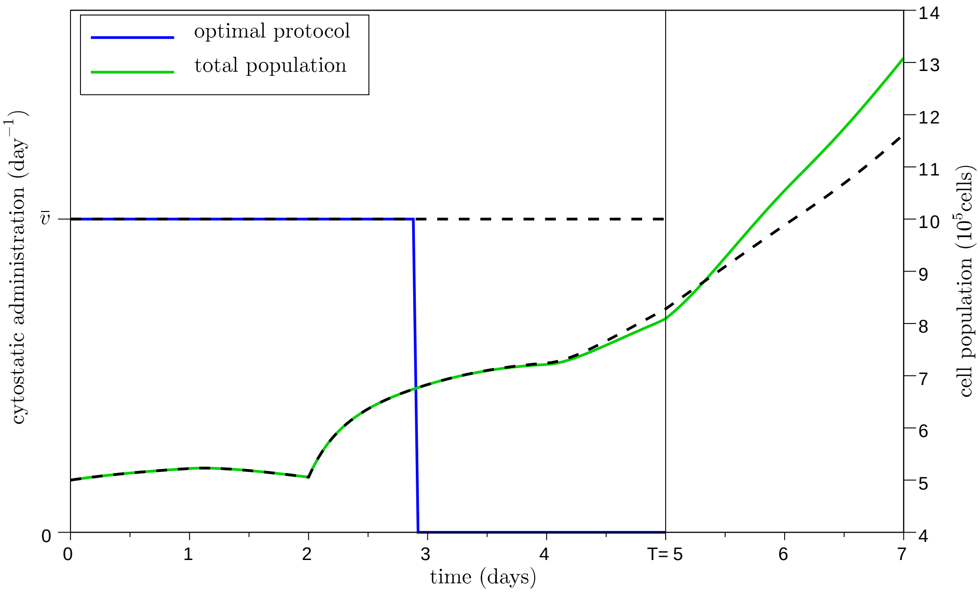

Figure 4.2: An horizon effect. We consider problem (4.39) with no cytotoxic u, $T = 5$ days, $\tau = 1$ day; the other parameters are given in Appendix 4.A.3 and are such that the proliferating phase globally produces cells. See Section 4.5.2 about numerical resolution. The solid lines (resp. the dash lines) represent the optimal protocol (resp. the $\bar{v}$-constant protocol) of cytostatic administration and the associated total population.

Resting cells and proliferating cells with different ages do not have the same role in the population dynamics. Thus it is natural not to give them the same weight in the objective function. One choice of age-dependent weight consists in the first dual eigenvector $(\bar{\Psi}, \bar{\phi})$, given by Theorem 4.3.7; it is justified by Remarks 4.3.9 and 4.3.11. After we stop administrating the drugs at time T, there is no action of the cytotoxic and a residual action of the cytostatic, as in Section 4.3.4; nothing can be done on the Malthus parameter λ, which is given by the uncontrolled system, but we can try to minimize the weighted total population

$$\bar{\Psi} R(\cdot) + \int_0^{2\tau} \bar{\phi}(a) p(\cdot, a) \mathrm{d}a \tag{4.40}$$

at time T. If there was no more action of the cytostatic after T, the weighted total population (4.40) would be constant for $t > T$ and would give the asymptotic size of the population (Theorem 4.3.8). It is not exactly the case with the residual action of the cytostatic (Lemma 4.3.10), but even though we choose this weighted total population at time T as the objective function.

4.4.2 Maximal cumulative doses

In order to limit the toxicity of the protocols, it is useful to add constraints on the cumulative doses of the drugs. Namely, we fix $\bar{U}, \bar{V}$ and we restrict the optimization problem to the protocols (u, v) such that

$$\int_0^T u(t)\mathrm{d}t \leq \bar{U}, \quad \int_0^T v(t)\mathrm{d}t \leq \bar{V}. \tag{4.41}$$

Note that with the bounds (4.38) on the controls, the constraints (4.41) are nontrivial iff

$$0 < \bar{U} < \bar{u}T, \quad 0 < \bar{V} < \bar{v}T,$$

respectively.

4.4.3 Reduction to a problem with delays

The state of the cell culture at the beginning of the experiments is fixed; it furnishes the initial condition $(k_0, R_0, p_0) \in \mathbb{R} \times \mathbb{R} \times L^\infty(0, 2\tau)$ of (4.1)-(4.4):

$$k(0) = k_0, \quad R(0) = R_0, \quad p(0, \cdot) = p_0 \tag{4.42}$$

with $k_0 = 0$, $R_0, p_0 \geq 0$. The issue of finding good protocols of drugs administration can finally be formulated as the following optimal control problem:

$$\begin{aligned} &\min_{(u,v,R,p,k)} \left(R(T) + \int_0^{2\tau} \bar{\Psi}^{-1}\bar{\phi}(a)p(T,a)\mathrm{d}a \right) \\ &\text{subject to} \quad \text{(4.1)-(4.4)}, \text{(4.38)}, \text{(4.41)-(4.42)}. \end{aligned} \tag{4.43}$$

Recall that $(\bar{\Psi}, \bar{\phi})$ is the first dual eigenvector, defined by Theorem 4.3.7.

Similarly to the derivation of Mackey's model (Section 4.2.4), (4.43) can be reduced to an optimal control problem of delay differential equations. We denote by $\tilde{p}$, $\tilde{P}$ and $\tilde{P}_2$ the weighted proliferating population density, the total weighted proliferating population and sub-population in the second-half of the phase, respectively:

$$\begin{gathered} \tilde{p}(t,a) := \bar{\Psi}^{-1}\bar{\phi}(a)p(t,a), \\ \tilde{P}(t) := \int_0^{2\tau} \tilde{p}(t,a)\mathrm{d}a, \quad \tilde{P}_2(t) := \int_\tau^{2\tau} \tilde{p}(t,a)\mathrm{d}a. \end{gathered}$$

Let (u, v, R, p, k) be such that (4.1)-(4.4),(4.42) hold. Observe that, along the characteristics a.e., there holds

$$\frac{\partial \tilde{p}}{\partial t}(t,a) + \frac{\partial \tilde{p}}{\partial a}(t,a) = (\lambda - \chi_{(\tau,2\tau)}(a)u(t))\tilde{p}(t,a).$$

Then by Lemma 4.3.2, there holds a.e.

$$\frac{\mathrm{d}R}{\mathrm{d}t}(t) = -(1-k(t))\beta R(t) + \tilde{p}(t,2\tau) \tag{4.44}$$

$$\frac{\mathrm{d}\tilde{P}}{\mathrm{d}t}(t) = \lambda\tilde{P}(t) - u(t)\tilde{P}_2(t) + \tilde{p}(t,0) - \tilde{p}(t,2\tau) \tag{4.45}$$

$$\frac{\mathrm{d}\tilde{P}_2}{\mathrm{d}t}(t) = (\lambda - u(t))\tilde{P}_2(t) + \tilde{p}(t,\tau) - \tilde{p}(t,2\tau) \tag{4.46}$$

$$\frac{\mathrm{d}k}{\mathrm{d}t}(t) = v(t)(1-k(t)) - \alpha k(t) \tag{4.47}$$

where, by definition of λ and by the method of characteristics,

$$\tilde{p}(t,0) = (1-k(t))(\lambda+\beta)R(t) \tag{4.48}$$

$$\tilde{p}(t,\tau) = \begin{cases} p_0(\tau-t)2e^{-\gamma t-(\lambda+\gamma)\tau} & \text{if } t<\tau \\ (1-k(t-\tau))(\lambda+\beta)R(t-\tau)e^{\lambda\tau} & \text{if } t>\tau \end{cases} \tag{4.49}$$

$$\tilde{p}(t,2\tau) = \begin{cases} p_0(2\tau-t)2e^{-\gamma t-y(t)} & \text{if } t<2\tau \\ (1-k(t-2\tau))(\lambda+\beta)R(t-2\tau)e^{\lambda 2\tau-y(t)} & \text{if } t>2\tau \end{cases} \tag{4.50}$$

$$\text{with} \quad y(t) := \begin{cases} \int_0^t u(s)\mathrm{d}s & \text{if } t<\tau \\ \int_{t-\tau}^t u(s)\mathrm{d}s & \text{if } t>\tau \end{cases}.$$

We consider y as a new state variable, and we also introduce two extra state variables U and V in order to handle the integral constraints (4.41):

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = \begin{cases} u(t) & \text{if } t<\tau \\ u(t)-u(t-\tau) & \text{if } t>\tau \end{cases}, \quad \frac{\mathrm{d}U}{\mathrm{d}t}(t) = u(t), \quad \frac{\mathrm{d}V}{\mathrm{d}t}(t) = v(t). \tag{4.51}$$

Observe that (4.44)-(4.51) is a system of ordinary differential equations for $t<\tau$; it becomes a system of differential equations with one discrete delay for $\tau<t<2\tau$, and with two discrete delays for $t>2\tau$. Its initial condition is the following:

$$\begin{gathered} R(0)=R_0, \quad k(0)=0, \quad y(0)=0, \quad U(0)=0, \quad V(0)=0, \\ \tilde{P}(0) = \int_0^{2\tau} 2e^{(\lambda+\gamma)(a-2\tau)}p_0(a)\mathrm{d}a, \quad \tilde{P}_2(0) = \int_\tau^{2\tau} 2e^{(\lambda+\gamma)(a-2\tau)}p_0(a)\mathrm{d}a. \end{gathered} \tag{4.52}$$

Problem (4.43) is therefore equivalent to the following optimal control problem:

$$\min_{(u,v,R,\tilde{P},\tilde{P}_2,k,y,U,V)} (R+\tilde{P})(T) \tag{4.53}$$

$$\text{subject to (4.44)-(4.52),} \quad \begin{cases} 0\le u(t)\le \bar{u} \\ 0\le v(t)\le \bar{v} \end{cases} \text{for a.a. } t\in(0,T), \text{ and } \begin{cases} U(T)\le \bar{U} \\ V(T)\le \bar{V} \end{cases}.$$

4.5 Results and conclusion

We present in this section some theoretical and numerical results on the optimal control problem introduced in the previous section. We use either its form (4.43) or (4.53); the data are

$$T>2\tau>0, \ R_0, p_0 \ge 0, \ \alpha,\beta>0, \ \gamma\ge 0, \ \bar{u},\bar{v},\bar{U},\bar{V}\ge 0.$$

The first eigenvalue λ is determined by (4.23).

4.5.1 Existence and optimality conditions

We begin with a result of existence of an optimal protocol of drugs administration. It relies on the fact that the dynamics is affine w.r.t. the controls. We do not have uniqueness in general.

Proposition 4.5.1. *There exists at least one optimal protocol of drugs administration $(\hat{u},\hat{v})$ with associated state $(\hat{R},\ldots,\hat{V})$.*

Proof. The value of problem (4.53) is non-negative; let $(u^k, v^k, R^k, \dots, V^k)$ be a minimizing sequence. Observe that (u^k, v^k) is bounded in L^∞, and $(R^k, \dots, V^k)$ is bounded and equicontinuous on $[0, T]$. Then by Banach-Alaoglu theorem and Arzelà-Ascoli theorem, there exists $(\hat{u}, \dots, \hat{V})$ such that, up to a subsequence,

$$(u^k, v^k) \rightharpoonup (\hat{u}, \hat{v}) \quad \text{and} \quad (R^k, \dots, V^k) \to (\hat{R}, \dots, \hat{V})$$

for the weak $*$ topology in L^∞ and the uniform topology in C^0, respectively. Since the dynamics is affine w.r.t. the controls, $(\hat{u}, \dots, \hat{V})$ satisfies (4.44)-(4.52). The bounds and the final constraints are also satisfied, and the objective function is minimized by construction. $\square$

The second result says that it is optimal to administrate as much of cytotoxic as possible. It implies uniqueness of the optimal protocol of cytotoxic administration when it is not constrained by a maximal cumulative dose. The constrained case will be studied numerically later.

Proposition 4.5.2. *Let $(\hat{u}, \hat{v})$ be an optimal protocol of drugs administration. Then*

$$\int_0^T \hat{u}(t)\mathrm{d}t = \min\left\{\bar{u}T, \bar{U}\right\}.$$

In particular, if $\bar{U} \geq \bar{u}T$, then $\hat{u}(t) = \bar{u}$ a.e. on $(0, T)$.

Proof. If $\hat{U}(T) < \min\left\{\bar{u}T, \bar{U}\right\}$, then there exists an admissible u such that $u \geq \hat{u}$, $u \neq \hat{u}$. The result follows from the fact that (4.2)-(4.4) is monotone w.r.t. $u \in L^\infty(0, T)$. $\square$

Next we state first-order optimality conditions, in the form of Pontryagin's minimum principle and where we highlight that the dynamics is affine w.r.t. the controls.

Proposition 4.5.3. *Let $(\hat{u}, \hat{v})$ be an optimal protocol of drugs administration. Then there exists $(a, b) \in W^{1,\infty}(0, T; \mathbb{R}^1)$ such that, for a.a. $t \in (0, T)$,*

$$(\hat{u}(t), \hat{v}(t)) \in \operatorname{argmin}\left\{a(t)u + b(t)v \ : \ \begin{array}{c} 0 \leq u \leq \bar{u} \\ 0 \leq v \leq \bar{v} \end{array}\right\}.$$

Proof. We apply Pontryagin's minimum principle to the delayed problem (4.53). It can be done either directly [50], or after Guinn's transformation [48, 49] into an optimal control problem of ordinary differential equations [18]. The minimized function is linear w.r.t. (u, y) at all time because the dynamics is affine w.r.t. the controls. See Appendices 4.A.1 and 4.A.2 for the precise statement of Pontryagin's minimum principle and the expression of coefficients a and b. $\square$

Then we expect, in the sense of the following corollary, the optimal protocols to be *bang-bang*, i.e. on their bounds.

Corollary 4.5.4. *For a.a. $t \in (0, T)$,*

$$\hat{u}(t) = \begin{cases} 0 & \text{if } a(t) > 0 \\ \bar{u} & \text{if } a(t) < 0 \end{cases} \quad \text{and} \quad \hat{v}(t) = \begin{cases} 0 & \text{if } b(t) > 0 \\ \bar{v} & \text{if } b(t) < 0 \end{cases}.$$

It is sometimes possible to determine the sign of b, and then the value of $\hat{v}$.

Proposition 4.5.5. *Let $\bar{V} \geq \bar{v}T$ and let $(\hat{u}, \hat{v})$ be an optimal protocol of drugs administration. Then there exists $\varepsilon > 0$ such that*

$$\hat{v}(t) = \begin{cases} 0 & \textit{a.e. on } (T-\varepsilon, T) \textit{ if } \lambda < 0 \\ \bar{v} & \textit{a.e. on } (T-\varepsilon, T) \textit{ if } \lambda > 0 \end{cases}.$$

Proof. It is important here to have Pontryagin's minimum principle with normal multipliers. See Appendix 4.A.2 for the determination of the sign of b. ☐

4.5.2 Optimal protocols

We use BOCOP [23] to solve numerically the undelayed optimal control problem obtained by Guinn's transformation [48, 49] of the delayed problem (4.53). We discuss here the optimal protocol $(\hat{u}, \hat{v})$, with associated state $(\hat{R}, \ldots, \hat{V})$, found numerically in different situations; we define the *minimal and maximal proliferating phase balances* respectively by

$$\delta_{\bar{u}} := 2e^{-(\gamma 2\tau + \bar{u}\tau)} - 1, \quad \delta_0 := 2e^{-\gamma 2\tau} - 1.$$

Note that $\delta_{\bar{u}} \leq \delta_0$, the latter having the same sign as λ by (4.23).

The case $\delta_0 \leq 0$ corresponds to a situation where the proliferating phase globally kills cells, even without the administration of any cytotoxic; $\lambda \leq 0$ and then there is no natural growth of the leukemic cell population: this is not a cancer situation. It could be seen that in this case, it is optimal to give no cytostatic: $\hat{v}(t) = 0$ a.e. on $(0, T)$, because the higher the global introduction rate, the greater the loss of cells.

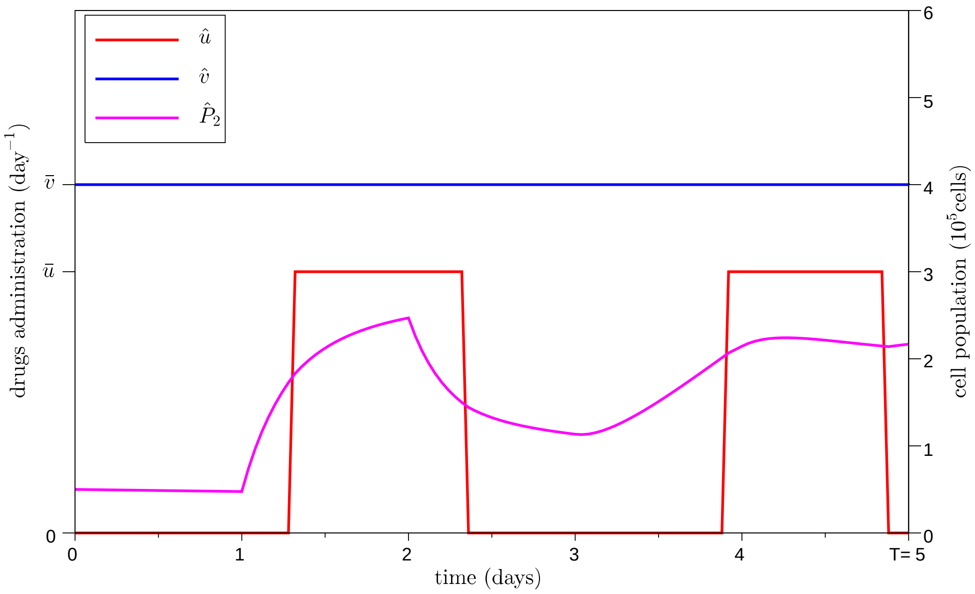

Figure 4.3: An optimal protocol with $0 \leq \delta_{\bar{u}}$ and limited cytotoxic. We consider a maximal cumulative dose of cytotoxic $\bar{U} = 2$ days$\cdot\bar{u}$, whereas $T = 5$ days, $\tau = 1$ day; the other parameters are given in Appendix 4.A.3. In addition to the optimal protocol of drugs administration $(\hat{u}, \hat{v})$, the associated total sub-population $\hat{P}_2$ is plotted.

The case $0 \leq \delta_{\bar{u}}$ corresponds to a situation where the proliferating phase globally produces cells, even with the administration of a maximum of cytotoxic; this is a very severe cancer situation. By Lemma 4.5.2, it is optimal to administrate as much of cytotoxic as possible; therefore we consider a nontrivial constraint (4.41) on the cumulative dose of cytotoxic with $0 < \bar{U} < \bar{u}T$, and no constraint on the cytostatic. We observe in Figure 4.3 that the optimal protocol of cytotoxic administration is bang-bang, with $\hat{u}(t) = \bar{u}$ a.e. when the total sub-population $\hat{P}_2$ (on which the cytotoxic is acting) is relatively high. And contrary to Section 4.4.1 and Figure 4.2, the optimal protocol of cytostatic administration is now $\hat{v}(t) = \bar{v}$ a.e. on $(0, T)$, because the lower the global introduction rate, the smaller the gain of cells.

The case $\delta_{\bar{u}} < 0 < \delta_0$ corresponds to a situation where the proliferating phase globally produces cells in absence of drugs, and the administration of cytotoxic can make it globally kill cells; this is the most interesting situation.

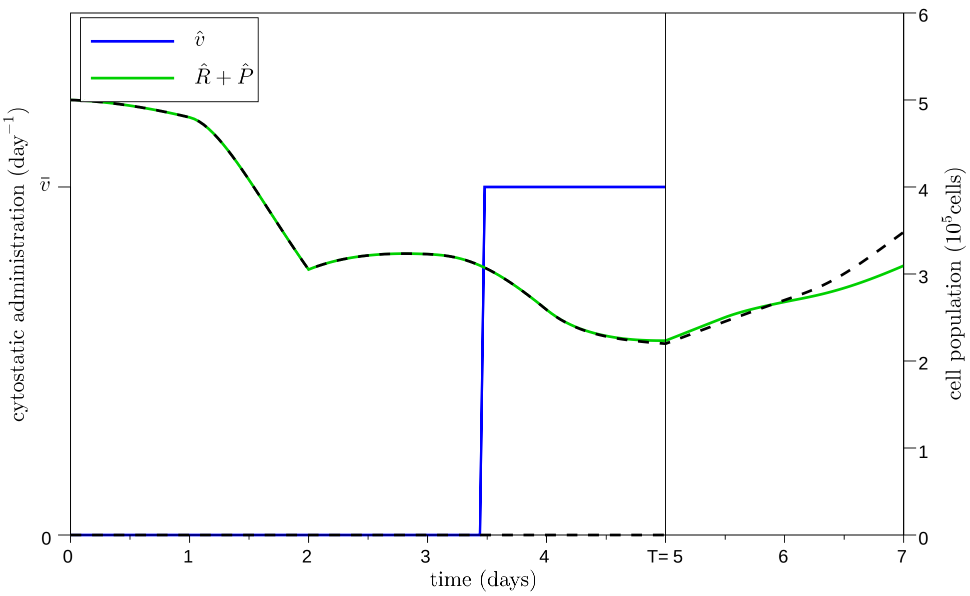

Figure 4.4: An optimal protocol of with $\delta_{\bar{u}} < 0 < \delta_0$. We consider no constraint on the cumulative dose of cytotoxic, $T = 5$ days, $\tau = 1$ day; the other parameters are given in Appendix 4.A.3. The protocol of cytotoxic administration is fixed to the optimal $\hat{u}(t) = \bar{u}$ a.e. on $(0, T)$ and is not plotted; the solid lines (resp. the dash lines) represent the optimal protocol (resp. the 0-constant protocol) of cytostatic administration and the associated total population.

First we consider no constraint on the cumulative doses of the drugs. The optimal protocol of cytotoxic administration is again $\hat{u}(t) = \bar{u}$ a.e. on $(0, T)$ and we do not plot it in Figure 4.4. We observe that the optimal protocol of cytostatic administration is bang-bang, with $\hat{v}(t) = 0$ a.e. first and $\hat{v}(t) = \bar{v}$ a.e. second. For comparison, we also plot the 0-constant protocol and the associated total population, which is slightly lower than for the optimal protocol at time T, but quickly becomes higher. The switch in the optimal protocol of cytostatic administration can be understood as follows: the resting cells which are introduced into the proliferating phase at time $t \in (0, T - 2\tau)$ will have a proliferating phase whose balance is $\delta_{\bar{u}} < 0$, whereas those introduced at time $t \in (T-\tau, T)$ will have a proliferating phase whose balance is $\delta_0 > 0$; it is therefore of interest to have a high global introduction rate during $(0, T - 2\tau)$ and a low one during $(T - \tau, T)$. Recall that we do not control directly the fraction of inhibited cells k, but the inhibition rate v. Note that by Proposition 4.5.5, we expected to have $\hat{v}(t) = \bar{v}$ a.e. at the end.

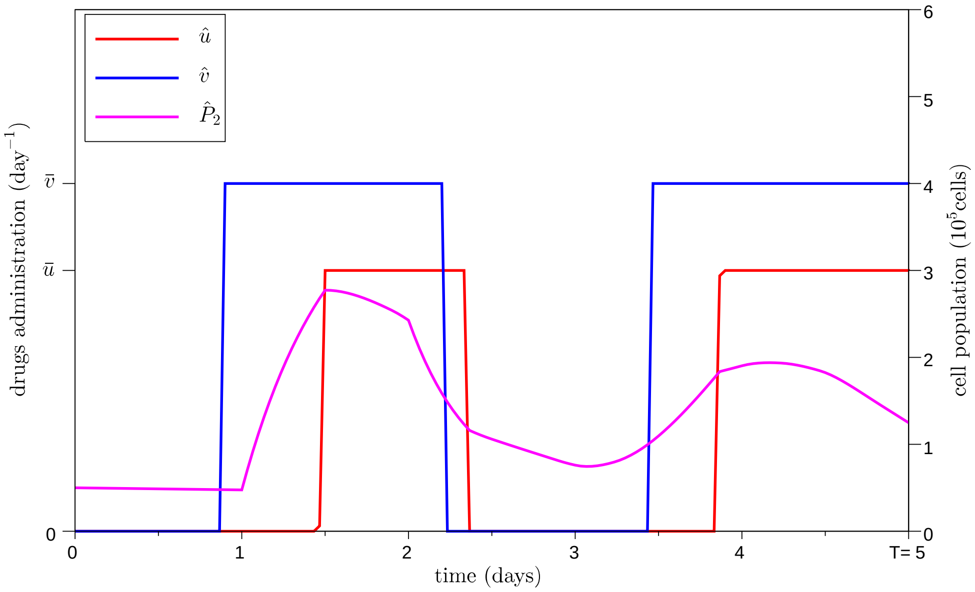

Figure 4.5: An optimal protocol with $\delta_{\bar{u}} < 0 < \delta_0$ and limited cytotoxic. We consider a maximal cumulative dose of cytotoxic $\bar{U} = 2$ days$\cdot\bar{u}$, whereas $T = 5$ days, $\tau = 1$ day; the other parameters are given in Appendix 4.A.3. In addition to the optimal protocol of drugs administration $(\hat{u}, \hat{v})$, the associated total sub-population $\hat{P}_2$ is plotted.

Second we add a nontrivial constraint on the cumulative dose of cytotoxic only, as in the case of Figure 4.3. Again, the optimal protocol of cytotoxic administration in Figure 4.5 is bang-bang, with $\hat{u}(t) = \bar{u}$ a.e. when the total sub-population $\hat{P}_2$ is relatively high. The optimal protocol of cytostatic administration $\hat{v}$ is also bang-bang, and its structure can be understood similarly to one of Figure 4.4: it is of interest to have a low global introduction rate if the cells which are just introduced are going to have a proliferating phase whose balance is positive, in particular if a time τ later, $\hat{u} = 0$ a.e. on a long enough interval; and it is also of interest to have a high global introduction rate a time τ before the intervals where $\hat{u} = \bar{u}$ a.e., in order for the cytotoxic to be efficient. Note that in this interpretation, $\hat{u}$ depends on $\hat{P}_2$, which depends on $\hat{v}$, which depends on $\hat{u}$.

4.5.3 Conclusion

The issue of finding good protocols of drugs administration in leukemic cell cultures has been formulated as an optimal control problem, where the population dynamics is eventually reduced to a delay differential system. It has to be noted that the definition of the objective function for this optimization problem is a nontrivial part of the modeling, and that it might still be improved if we could find explicitly the limit I_∞ in Lemma 4.3.10.

This approach is different from [14], where the objective function is the Floquet eigenvalue of a periodic problem.

A few optimal protocols have been presented to illustrate different behaviors, which are not always intuitive. Optimal protocols in general have not been synthesized; the dimension 7 of the differential system and the fact that the adjoint state equations are with advanced arguments have to be added to the complexity described in [59] for combined treatments. Nevertheless, it is not excluded to get further results on bang-bang and singular controls, as in [59, 60], from the analysis started in Appendix 4.A.2.

Estimated parameters are needed for medical application and are to be published [9]. The optimal control problem being set and numerically implemented, it could suggest in vitro protocols to the biologists, and maybe answer questions of the clinicians. It could also simulate experiments longuer than 5 days, which are complicated to carry out for pratical reasons. For in vivo modeling, pharmacokinetic-pharmacodynamic (PK-PD) would have to be added, as in [10, 59, 60].

4.A Appendix

4.A.1 Pontryagin's minimum principle

We consider problem (4.53) and its dynamics (4.44)-(4.51). We denote by R^0, R^1, and R^2 the arguments of the state variable R with a delay 0, τ, and 2τ, respectively; we denote similarly the different arguments of all the undelayed and delayed state and control variables. We then define the following functions of t and

$$(u^0, u^1, v^0, R^0, R^1, R^2, \tilde{P}^0, \tilde{P}_2^0, k^0, k^1, k^2, y^0, U^0, V^0)$$

by

$$F_R(t,\cdot) := \begin{cases} -(1-k^0)\beta R^0 + p_0(2\tau - t)2e^{-\gamma t - y^0} & \text{if } t < 2\tau \\ -(1-k^0)\beta R^0 + (1-k^2)(\lambda+\beta)R^2 e^{\lambda 2\tau - y^0} & \text{if } t > 2\tau \end{cases}$$

$$F_{\tilde{P}}(t,\cdot) := \begin{cases} \lambda\tilde{P}^0 - u^0\tilde{P}_2^0 + (1-k^0)(\lambda+\beta)R^0 & \\ \qquad - p_0(2\tau - t)2e^{-\gamma t - y^0} & \text{if } t < 2\tau \\ \lambda\tilde{P}^0 - u^0\tilde{P}_2^0 + (1-k^0)(\lambda+\beta)R^0 & \\ \qquad -(1-k^2)(\lambda+\beta)R^2 e^{\lambda 2\tau - y^0} & \text{if } t > 2\tau \end{cases}$$

$$F_{\tilde{P}_2}(t,\cdot) := \begin{cases} (\lambda - u^0)\tilde{P}_2^0 + p_0(\tau - t)2e^{-\gamma t - (\lambda+\gamma)\tau} & \\ \qquad - p_0(2\tau - t)2e^{-\gamma t - y^0} & \text{if } t < \tau \\ (\lambda - u^0)\tilde{P}_2^0 + (1-k^1)(\lambda+\beta)R^1 e^{\lambda\tau} & \\ \qquad - p_0(2\tau - t)2e^{-\gamma t - y^0} & \text{if } \tau < t < 2\tau \\ (\lambda - u^0)\tilde{P}_2^0 + (1-k^1)(\lambda+\beta)R^1 e^{\lambda\tau} & \\ \qquad -(1-k^2)(\lambda+\beta)R^2 e^{\lambda 2\tau - y^0} & \text{if } t > 2\tau \end{cases}$$

$$F_k(t,\cdot) := v^0(1-k_1) - \alpha k^0 \qquad F_y(t,\cdot) := \begin{cases} u^0 & \text{if } t < \tau \\ u^0 - u^1 & \text{if } t > \tau \end{cases}$$

$$F_U(t,\cdot) := u^0 \qquad F_V(t,\cdot) := v^0$$

The system (4.44)-(4.51) can now be written as

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t) = F_x(t, u(t), u(t-\tau), v(t), R(t), R(t-\tau), R(t-2\tau), \tilde{P}(t), \tilde{P}_2(t),$$
$$k(t), k(t-\tau), k(t-2\tau), y(t), U(t), V(t))$$

for $x \in \{R, \tilde{P}, \tilde{P}_2, k, y, U, V\}$ and for a.a. $t \in (0,T)$.

We define the *Hamiltonian* and the *final point Lagrangian* respectively as follows: given $q = (q_R, q_{\tilde{P}}, q_{\tilde{P}_2}, q_k, q_y, q_U, q_V) \in W^{1,\infty}(0,T;\mathbb{R}^7)$, let

$$H[q](t,\cdot) := \sum_x q_x(t) F_x(t,\cdot), \quad x \in \{R, \tilde{P}, \tilde{P}_2, k, y, U, V\},$$

and given $\Psi = (\Psi_U, \Psi_V) \in \mathbb{R}^2$, let

$$\Phi[\Psi](\cdot) := R^0 + \tilde{P}^0 + \Psi_U(U^0 - \bar{U}) + \Psi_V(V^0 - \bar{V}),$$

where $\cdot$ stands again for $(u^0, u^1, v^0, R^0, R^1, R^2, \tilde{P}^0, \tilde{P}_2^0, k^0, k^1, k^2, y^0, U^0, V^0)$.

Given a protocol of drugs administration $(\hat{u}, \hat{v})$, and $(\hat{R}, \dots, \hat{V})$ its associated state, we denote by $\widehat{H}[q](t)$ the evalution of $H[q]$ at

$$(t, \hat{u}(t), \hat{u}(t-\tau), \hat{v}(t), \hat{R}(t), \hat{R}(t-\tau), \hat{R}(t-2\tau), \dots, \hat{V}(t)),$$

by $\widehat{\Phi}[\Psi](T)$ the evaluation of $\Phi[\Psi]$ at

$$(\hat{u}(T), \hat{u}(T-\tau), \hat{v}(T), \hat{R}(T), \hat{R}(T-\tau), \hat{R}(T-2\tau), \dots, \hat{V}(T)),$$

and similarly for their partial derivatives. We can now state Pontryagin's principle:

Theorem 4.A.1. *Let $(\hat{u}, \hat{v})$ be an optimal protocol of drugs administration with associated state $(\hat{R}, \dots, \hat{V})$. Then there exist $q \in W^{1,\infty}(0,T;\mathbb{R}^7)$ and $\Psi \in \mathbb{R}^2$ such that, for a.a. $t \in (0,T)$,*

$$-\frac{\mathrm{d}q_x}{\mathrm{d}t}(t) = D_{x^0}\widehat{H}[q](t) + \chi_{(0,T-\tau)}(t) D_{x^1}\widehat{H}[q](t+\tau) \qquad q_x(T) = D_{x^0}\widehat{\Phi}[\Psi](T)$$
$$+ \chi_{(0,T-2\tau)}(t) D_{x^2}\widehat{H}[q](t+2\tau), \tag{4.54}$$

for $x \in \{R, \tilde{P}, \tilde{P}_2, k, y, U, V\}$; for a.a. $t \in (0,T)$,

$$\widehat{H}[q](t) + \chi_{(0,T-\tau)}(t)\widehat{H}[q](t+\tau) \le H[q](t, u, \hat{u}(t-\tau), v, \hat{R}(t), \dots, \hat{V}(t))$$
$$+ \chi_{(0,T-\tau)}(t) H[q](t+\tau, \hat{u}(t+\tau), u, \hat{v}(t+\tau), \hat{R}(t+\tau), \dots, \hat{V}(t+\tau)) \tag{4.55}$$

for all $(u,v) \in [0,\bar{U}] \times [0,\bar{V}]$; and

$$\Psi_U \ge 0, \quad \Psi_U(\hat{U}(T) - \bar{U}) = 0,$$
$$\Psi_V \ge 0, \quad \Psi_V(\hat{V}(T) - \bar{V}) = 0.$$

Proof. This is Pontryagin's minimum principle [18, 48, 50]. Observe that problem (4.53) satisfies a Mangasarian-Fromovitz condition of qualification if $\bar{U}, \bar{V} > 0$; we consider the optimal control problem without the control u (resp. v) if $\bar{U} = 0$ (resp. $\bar{V} = 0$), and it becomes qualified. Then we get the existence of normal multipliers [18]. ∎

4.A.2 Proof of Propositions 4.5.3 and 4.5.5

By (4.54), $q_U \equiv \Psi_U$ and $q_V \equiv \Psi_V$. Since $H[q]$ is affine w.r.t. (u^0, u^1, v^0), we derive Proposition 4.5.3 from the Hamiltonian minimum condition (4.55), with

$$\begin{aligned}
a(t) &= D_{u^0}\widehat{H}[q](t) + \chi_{(0,T-\tau)}(t)D_{u^1}\widehat{H}[q](t+\tau) \\
&= \Psi_U - (q_{\tilde{P}}(t) + q_{\tilde{P}_2}(t))\hat{\tilde{P}}_2(t) + q_y(t) + \chi_{(0,T-\tau)}(t)q_y(t+\tau), \\
b(t) &= D_{v^0}\widehat{H}[q](t) \\
&= \Psi_V + q_k(t)(1-\hat{k}(t)).
\end{aligned}$$

For Proposition 4.5.5, we need to determine the sign of b in a neighborhood of T. Let $\bar{V} \geq \bar{v}T$; considering the equivalent optimization problem without the constraint on V, we can assume that $\Psi_V = 0$. Since $1 - \hat{k} > 0$, b has then the same sign as q_k, whose adjoint equation (4.54) is

$$\begin{aligned}
\frac{\mathrm{d}q_k}{\mathrm{d}t}(t) = {} & q_k(t)(\hat{v}(t)+\alpha) - (q_R - q_{\tilde{P}})(t)\beta\hat{R}(t) + q_{\tilde{P}}(t)\lambda\hat{R}(t) \\
& + \chi_{(0,T-\tau)}(t)q_{\tilde{P}_2}(t+\tau)(\lambda+\beta)\hat{R}(t)e^{\lambda\tau} \\
& + \chi_{(0,T-2\tau)}(t)(q_R - q_{\tilde{P}} - q_{\tilde{P}_2})(t+2\tau)(\lambda+\beta)\hat{R}(t)e^{\lambda 2\tau - \hat{y}(t+\tau)}
\end{aligned}$$

for a.a. $t \in (0,T)$, and $q_k(T) = 0$.

Lemma 4.A.2. *Let $c, d \in L^\infty(\mathbb{R})$ and $w, z \in W^{1,\infty}_{loc}$ be such that, for a.a. t,*

$$\begin{aligned}
\dot{w}(t) &= c(t)w(t) + d(t) & w(T) &= \bar{w}, \\
\dot{z}(t) &= d(t)e^{\int_t^T c(\theta)\mathrm{d}\theta} & z(T) &= \bar{w}.
\end{aligned}$$

Then w and z have the same sign.

Proof. Simply observe that

$$\begin{aligned}
w(t) &= \left(\bar{w} - \int_t^T d(s)e^{\int_s^T c(\theta)\mathrm{d}\theta}\mathrm{d}s\right)e^{-\int_t^T c(\theta)\mathrm{d}\theta} \\
&= z(t)e^{-\int_t^T c(\theta)\mathrm{d}\theta}.
\end{aligned}$$

□

Let $f \in L^\infty(0,T)$ be defined by

$$\begin{aligned}
f(t) := {} & -(q_R - q_{\tilde{P}})(t)\beta + q_{\tilde{P}}(t)\lambda + \chi_{(0,T-\tau)}(t)q_{\tilde{P}_2}(t+\tau)(\lambda+\beta)e^{\lambda\tau} \\
& + \chi_{(0,T-2\tau)}(t)(q_R - q_{\tilde{P}} - q_{\tilde{P}_2})(t+2\tau)(\lambda+\beta)e^{\lambda 2\tau - \hat{y}(t+\tau)}
\end{aligned}$$

and $\sigma \in W^{1,\infty}(0,T)$ be such that, for a.a. t,

$$\dot{\sigma}(t) = f(t)\hat{R}(t)e^{\int_t^T(\hat{v}(\theta)+\alpha)\mathrm{d}\theta} \qquad \sigma(T) = 0.$$

Then b has the same sign as σ. By the final condition of the adjoint equations (4.54), $f(T) = \lambda$. Since f is left-continuous on T, there exists $\varepsilon > 0$ such that f, and then $\dot{\sigma}$, have the same sign as λ on $(T-\varepsilon, T)$. Proposition 4.5.5 follows.

4.A.3 Parameters for the numerical resolutions

These parameters have not been estimated; some of them are fixed in coherence with data from the experiments described in the introduction [9], the others are chosen to explore different situations.

Figure 4.2 The parameters are the following:

$$T = 5 \text{ days} \quad \tau = 1 \text{ day} \tag{4.56}$$

$$R_0 = 4 \times 10^5 \text{ cells} \quad p_0(a) = 0.5 \times 10^5 \text{ cells} \times \text{ day}^{-1} \tag{4.57}$$

$$\alpha = 1 \text{ day}^{-1} \quad \beta = 2 \text{ day}^{-1} \quad \bar{v} = 2 \text{ day}^{-1} \tag{4.58}$$

$$\gamma = 0.15 \text{ day}^{-1} \tag{4.59}$$

The proliferating phase balance is then $2e^{-\gamma 2\tau} - 1 \approx 0.48 > 0$.

Figure 4.3 The parameters are the following: (4.56)-(4.58) and

$$\gamma = 0.05 \text{ day}^{-1} \quad \bar{u} = 0.2 \text{ day}^{-1} \quad \bar{U} = 2 \text{ days} \cdot \bar{u} \tag{4.60}$$

Note that for these values, $0 < \delta_{\bar{u}} \approx 0.48 < \delta_0 \approx 0.81$. Solving numerically (4.23), we get $\lambda \approx 0.24 \text{ day}^{-1} > 0$.

Figure 4.4 The parameters are the following: (4.56)-(4.58) and

$$\gamma = 0.05 \text{ day}^{-1} \quad \bar{u} = 1 \text{ day}^{-1} \tag{4.61}$$

Note that now, $\delta_{\bar{u}} \approx -0.33 < 0$.

Figure 4.5 The parameters are the following: (4.56)-(4.58),(4.61) and

$$\bar{U} = 2 \text{ days} \cdot \bar{u} \tag{4.62}$$

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